Title:
Application Perspectives of Induction Surfaces in CAD: Smooth Connection of Concentric Spherical Surfaces

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Introduction:

Induction surfaces were recently introduced to observe the effect of linear transformations (multiplication by matrices) on the norm of vectors in more detail [4]. Specifically, they visualize for each spatial direction the multiplication factor of the result vector's norm compared to the original vector. In planar setting one would arrive at induction curves, and in a general n-dimensional setting induction sets, induction manifolds were defined. The aesthetics and peculiar properties of induction curves and surfaces provide them with good potential for future application in Computer-Aided Geometric Design. The presence of so-called p-eigenvectors and p-eigensubspaces further enrich their interesting properties [5, 6].

In the present work, after giving the definition and some examples for such geometric objects, we explore the topic of connecting segments of concentric spherical surfaces in a smooth manner. We show that by using the maximum norm (as the p-norm in the definition of induction surfaces) and applying diagonal matrices we immediately arrive at concentric spherical sections of different radii with continuous connection in-between. Lowering the norm to a finite but considerably large value we get a differentiable connection, sacrificing however the exactness of the spherical subsurfaces. This smoothing effect is similar to that of Catmull–Clark and Doo–Sabin subdivision surfaces, but here we have explicit surfaces, not meshes, nor splines.

The artistic possibilities are also to be explored further, similarly as seen in the planar case for Pythagorean hodograph curves in Klár–Valasek's [3]. Application possibilities of the current exploration includes visualizing solid spherical objects with emphasis on a peculiar set of layers, such as in geology (structure of the Earth) or physics.

The Definition of Induction Curves and Surfaces:

The p-norms or power norms for vectors are used as in general with $2 \le n \in \mathbb{N}$ as follows:

$$\|.\|_p : \mathbb{R}^n \to \mathbb{R}, \qquad \|x\|_p = \left(\sum_{k=1}^{n} |x_k|^p \right)^{1/p} \qquad (p \in [1, \infty)),$$

and

$$\|x\|_\infty = \max_{k=1}^{n} |x_k|.$$

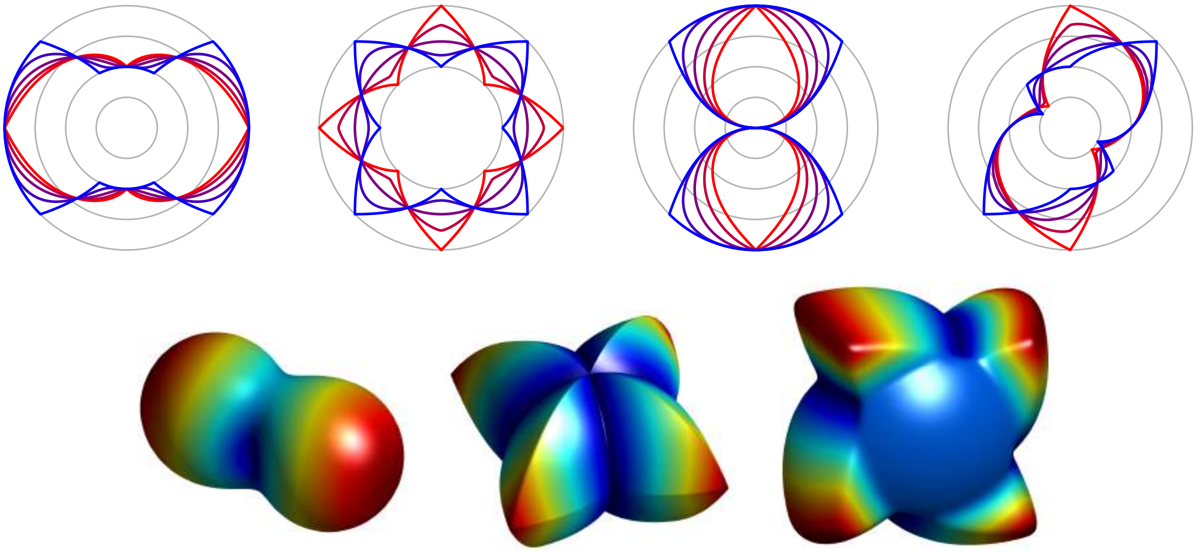

Fig. 1: Top row. Some examples for induction curves, from left to right: diagonal matrix, rotation matrix, singular diagonal matrix, and skew matrix. Gray circles denote the radial units, red to blue colors correspond to varying p values with 1, 4/3, 2, 4, $+\infty$. Bottom row. Some induction surfaces for various matrices and p values with code names "peanut", "compass" and "propeller".

It is well known that $\lim_{p\to+\infty} \|x\|_p = \|x\|_\infty$ $(x \in \mathbb{R}^n)$. Let us now consider a matrix $A \in \mathbb{R}^{n\times n}$. The p-norm of A is defined as follows ("induced" by the vector p-norm):

$$\|.\|_p : \mathbb{R}^{n\times n} \to \mathbb{R}, \qquad \|A\|_p = \sup_{x\neq 0} \frac{\|Ax\|_p}{\|x\|_p} \qquad (p \in [1,\infty]) .$$

As an equivalent description one may consider the supremum only for the elements of the unit sphere, i.e. $\|x\|_p = 1$, thus the fraction becomes unnecessary. In this work, we restrict ourselves for $n = 2$ and 3.

Let us now define *induction sets* in general, with $n = 2$ resulting in *induction curves* and $n = 3$ in *induction surfaces*. Given a matrix $A \in \mathbb{R}^{n\times n}$ with $2 \le n \in \mathbb{N}$ and $p \in [1,\infty]$, the set of points

$$\mathcal{I}_p(A) := \left\{ \frac{\|Ax\|_p}{\|x\|_p} \cdot \frac{x}{\|x\|_2} \in \mathbb{R}^n \ : \ 0 \neq x \in \mathbb{R}^n \right\} \subset \mathbb{R}^n$$

is called the *induction set* of A with parameter p. Instead of all (but the zero) vectors we may consider only one vector for each direction (e.g. the one with norm 1) since the expression $x/\|x\|_2$ is present in the definition. And thus the formula tells us to examine for each direction the fraction $\|Ax\|_p / \|x\|_p$, i.e. what is the multiplication factor for the change in the norm after multiplying our vector with the matrix.

Finally, for each direction we will have one point on a continuous curve or surface around the origin (for regular matrices). It is easy to see that we get a symmetric object with respect to the origin. For a more detailed description and properties we refer to [4].

Fig. 1 presents some (sets of) examples for induction curves and induction surfaces. Matlab programs were used to generate these images, available under https://locsi.web.elte.hu/indsets/.

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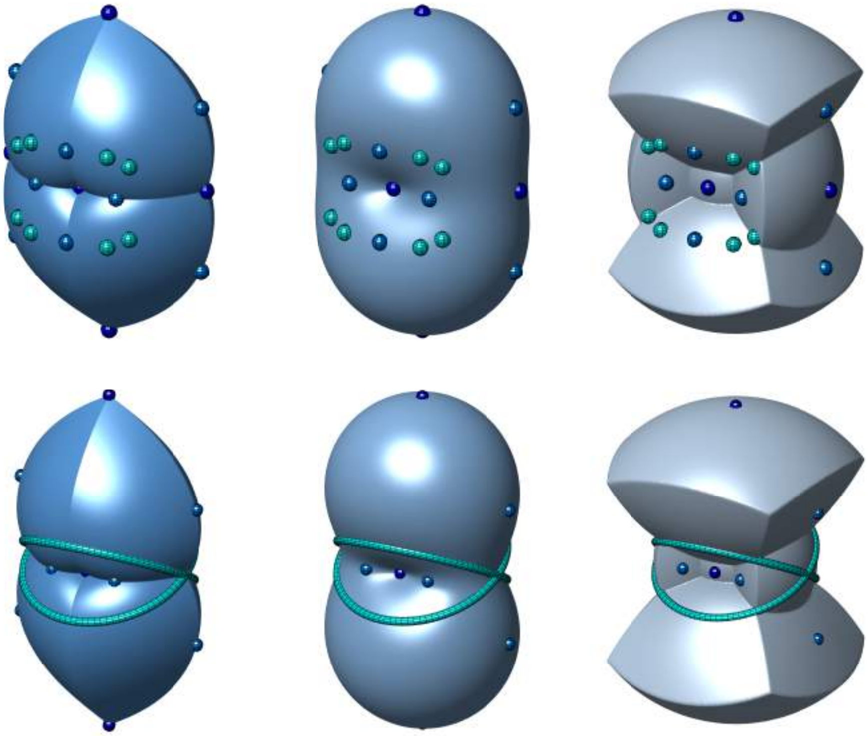

Fig. 2: Induction surfaces for two diagonal matrices (rows) with three different p-norms (columns). The common intersection points and curves of the surfaces row-wise are marked.

Induction Surfaces of Diagonal Matrices, with p-eigenvectors:

We highlight here two more sets of images of induction surfaces with our purpose being twofold. On one hand, this showcases an interesting property of the given surfaces, related to p-eigenvectors, on the other hand, some of these images serve as primary motivation for our current exploration.

Fig. 2 shows 3+3 induction surfaces for the diagonal matrices $\mathrm{diag}(1, 2, 3)$ and $\mathrm{diag}(1, 2, 4)$ with norms $p = 1,\ 2,\ +\infty$. Although only the surfaces with these three norms are presented, any p value in the interval $[1, +\infty]$ could have been used. (Actually also between 0 and 1.) It is encouraged to imagine now the smooth transition between these surfaces as p visits $[1, +\infty]$ with the "key frames" as depicted. A similar transition for the induction curves on Fig. 1 (red to blue, for all four sets of curves) may be considered.

Varying the parameter p as described above for fixed matrices led to the discovery of p-eigenvectors. Visually these are points that are common intersections of all the infinitely many curves/surfaces depending on $p \in [1, +\infty]$. These common intersection points corresponding to p-eigendirections can be observed in case of the induction curves of Fig. 1, and are marked with small spheres on Fig. 2. In case of some special matrices the constellation of these directions blends to form a p-eigensubspace, visually one (or rather two) common intersection curve(s) of all infinitely many surfaces. For a more detailed treatment of this topic see [4, 5].

Now we are motivated by the surfaces (and curves) for diagonal matrices with $p = +\infty$. As it can be seen—and was also confirmed mathematically—these include spherical partial surfaces (circular partial curves). These are concentric, and the radii are exactly the diagonal elements of the matrix. Furthermore, these segments are connected in a continuous manner.

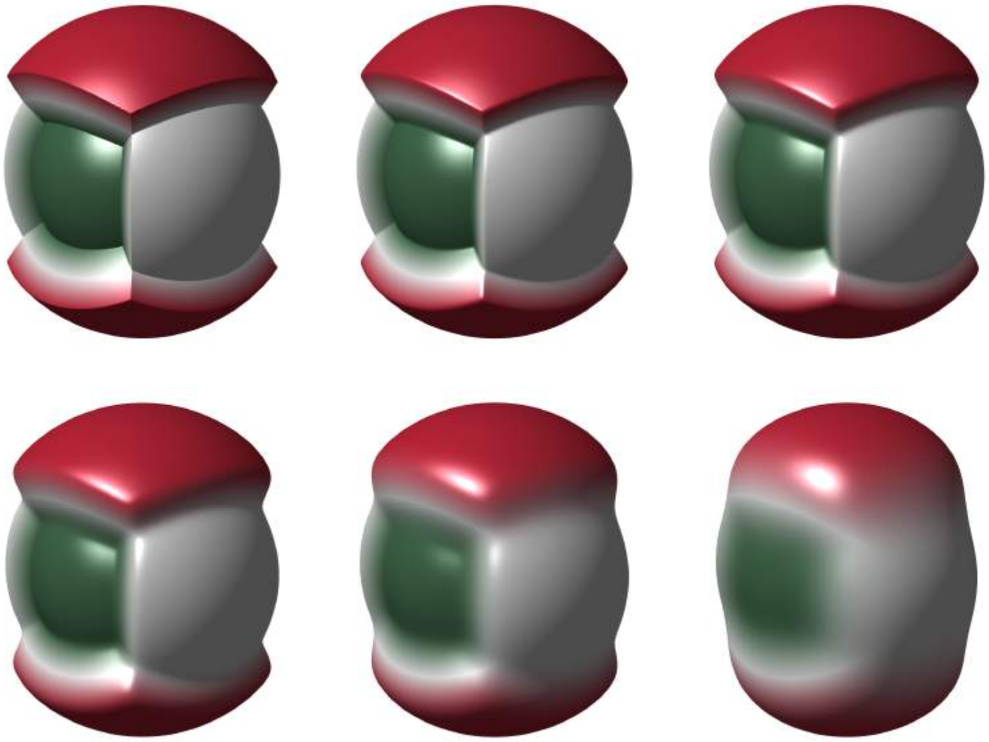

Fig. 3: Continuous and smoothed connection of concentric spherical segments as given by induction surfaces of diagonal matrices with decreasing p values. Colors indicate the distances from the origin.

Smooth Connection of Concentric Spherical Surfaces:

As seen in the previous section, induction surfaces of diagonal matrices with $p = +\infty$ norm give us spherical surface segments with a continuous connection in between. Applying norms with high p values will smooth our surface at hand, providing a differentiable transition between the spherical segments, sacrificing however the exactness of the original spherical parts.

Fig. 3 illustrates this smoothing process in case of the diagonal matrix $\mathrm{diag}(4, 5, 6)$. We start off with the "original" surface with $p = +\infty$ having exact spherical segments (with radii 4, 5 and 6) and their continuous connection, then the subsequent images reduce the value of p to the power-of-two values 64, 32, 16, 8 and 4. The distance of the points from the origin is indicated by colors. Now the colors of the national flag of Hungary (red, white, green) were chosen to honor the location of the CAD'24 conference.

One may observe the gradually more smooth surfaces, but also the decay of the accuracy of the exact spherical sections. Although the colors suggest that the original radii are mostly kept inside the regions of interest. Note that the exact radii are theoretically only preserved along the axes. The appropriate parameter value (for smoothness) may be chosen depending on the application's purpose.

Naturally, if two elements of the diagonal matrix are the same, then we will have an induction surface composed basically of two concentric spheres with two different radii. If all three elements were the same, then we would arrive at a sphere—independent of the norm—which is now of negligible interest.

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Criticism and Further Work:

Although induction surfaces provide an elegant way to connect concentric spherical surfaces, the location of these spherical parts were also provided by the induction sets at hand, depending on the initial diagonal matrix. One may observe that a bigger radius gives rise to a bigger surface with this radius, and vice-versa, smaller segments are present with smaller radius. For an application perspective, it would be desired that the location and size of the spherical surfaces could be also prescribed by the user and then connect the given parts. A direction of further research is to describe a way to solve this problem. Our current idea would be to use a so-called argument transformation, some suitable bijection of the unit sphere onto itself (or a group of these with respect to composition). The loss of precision in the spherical sections may be also further examined and quantified.

Of course, there are many further questions posed already regarding induction sets and p-eigenvectors not directly related to CAD [4, 5, 6].

Conclusions:

The potential application areas of induction curves and induction surfaces in Computer-Aided Design are yet unexplored. However, these explicit surfaces may prove useful for the solution of some specific problems. Also, their aesthetics and special properties make them worth studying for the design of visual elements.

In this work, we show that induction surfaces may be used to provide continuous and even smoothed connection between concentric spherical segments. The introduced error in the spherical regions may be analyzed in more detail. Furthermore, the problem of connecting user-specified concentric spherical sections (or at least give an appropriate approximation) is yet to be examined.

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References:

[1] Catmull, E.; Clark, J.: Recursively generated B-spline surfaces on arbitrary topological meshes, Computer Aided Design, 10(6), 1978, 350–355. https://doi.org/10.1016/0010-4485(78)90110-0

[2] Doo, D.; Sabin, M.: Behaviour of recursive division surfaces near extraordinary points, Computer Aided Design, 10(6), 1978, 356–360. https://doi.org/10.1016/0010-4485(78)90111-2

[3] Klár, G.; Valasek, G.: Employing Pythagorean hodograph curves for artistic patterns, Acta Cybernet., 20, 2011, 101–110. https://doi.org/10.14232/actacyb.20.1.2011.8

[4] Lócsi, L.: Introducing p-eigenvectors, exact solutions for some simple matrices, Ann. Univ. Sci. Budapest. Sect. Comput. 49, 2019, 325–345.

[5] Lócsi, L.; Németh, Zs.: On the construction of p-eigenvectors, Ann. Univ. Sci. Budapest. Sect. Comput. 50, 2020, 231–247.

[6] Lócsi, L.: Identification of induction curves, Stud. Univ. Babeş–Bolyai Math., 68(3), 2023, 467–480. https://doi.org/10.24193/subbmath.2023.3.01