Title:
Generation of κ-Space Curve

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Introduction:

κ-curve [1] is a recently published interpolating spline which consists of quadratic Bézier segments passing through input points at the loci of local curvature extrema.

But there is still significant space for κ-curve to be improved. Such as their curve is almost curvature-continuous everywhere, except at inflection points only G^1 continuity is guaranteed, i.e. the absolute value of curvature around the joints is euqals to each other but the sign is reversed ; since the degree of freedom (DoF) of the quadratic segments is limited, it is impossible to control the magnitudes of local maximum curvature at the input points; since the quadratic Bézier curve is planar, they cannot represent 3D curve.

Wang et al. [2] solved the first shortcoming by using log-aesthetic curves instead of Bézier curve. Miura at al. [3] solved the second shortcoming by elevating the degree of the Bernstein functions. In this paper, we propose a new method to solve the last shortcoming by replacing the quadratic Bézier segments by quartic Bézier segments and maintaining G^2 continuity.

Generation of κ-Curve:

Yan et al. [1] create a sequence of quadratic curves with G^2 continuity almost everywhere and passes through input points at the local curvature extrema. In order to obtain G^1 continuity, as shown in Fig. 1 the control points $c_{i,2}$ and $c_{i+1,0}$ are set with the constant λ_i

$$c_{i,2} = c_{i+1,0} = (1-\lambda_i)c_{i,1} + \lambda_i c_{i+1,1} \tag{2.1}$$

Interpolation of the input points at the local curvature maximum is guaranteed by the condition

$$c_i(t_i) = p_i$$

and control points $c_{i,1}$ is obtained by

$$c_{i,1} = \frac{p_i - (1-t_i)^2 c_{i,0} - t_i^2 c_{i,2}}{2t_i(1-t_i)} \tag{2.2}$$

Proceedings of CAD'23, Mexico City, Mexico, July 10-12, 2023, 85-89

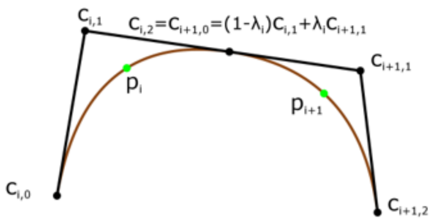


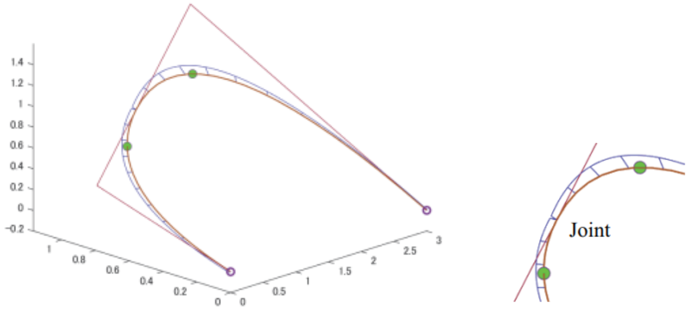
Fig. 1: Control points setting of κ-curve [1]

and the G^2 condition derives

$$\lambda_i = \frac{\sqrt{\triangle_i^+}}{\sqrt{\triangle_i^+} + \sqrt{\triangle_{i+1-}}} \tag{2.3}$$

introducing the notations $\triangle_i^+ = \triangle(c_{i,0}, c_{i,1}, c_{i+1,1})$ and $\triangle_{i+1}^- = \triangle(c_{i,1}, c_{i+1,1}, c_{i+1,2})$, and $\triangle$ represents area of the triangle.

At first we use quadratic Bézier curves to form κ-space curve and the contol points of a quadratic Bézier curve are on the same plane. Even though the control points are located in 3D space, Eqs. (2.1) and Eqs. (2.2) can be applicable for them. In Eq. (2.3), we need areas of $\triangle_i^+$ and $\triangle_i^-$, and the control points are on the same plane, respectivey. It is straightforwrad to generate a curve for a sequence of 3D input points as shown in Fig. 2. Since the planes where the control points of consecutive quadratic Bézier curves are located as well as their Frenet frames are generally different, G^2 continuity is broken at their joints although the absolute values of curvature are the same (please see the closeup in Fig. 2(b)).Only G^1 continuity is guarantted although the absolute values of curvature are the same.



(a) (b)

Fig. 2: κ-space open curve

Replacement with Quartic Bézier Curves:

For the κ-space curve introduced in the previous section, we would like to replace two consecutive quadratic Bézier curve with a Bézier curve of higher degree. In order to achieve G^2 continuity, the joint between two consecutive quadratic segments should be replaced with guaranteeing it at both ends of the replacing curve segment of higer degree. The three control points of these two quadratic segments

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generally form different planes each other as shown in Fig. 3 and we need a curve of at least degree four for G^2 continuity.

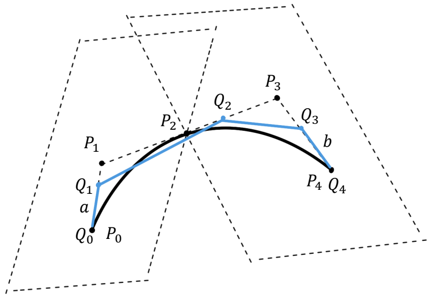


Fig. 3: Whole replacement control points setting.

Hence we use a quartic Bézier curve for replacement. The quartic Bézier curve has five control points and the first three control points should be on the plane of the first quadratic curve segment and the last three ones on that of the second curve segment. This means that the first and second control points are on the first plane, the fourth and fifth ones on the second plane and the third control points must be located on the line of P_1P_3 in Fig. 3. Furthermore for G^1 continuity at the ends, the second control point is on P_0P_1 and the third one on P_3P_4. Therefore for the control points $Q_i(i = 0, \cdots, 4)$ of the quartic curve, there exist such a, b and γ that

$$
\begin{aligned}
Q_0 &= P_0 \\
Q_1 &= (1-a)P_0 + aP_1 \\
Q_2 &= (1-\gamma)P_1 + \gamma P_3 \\
Q_3 &= (1-b)P_3 + bP_4 \\
Q_4 &= P_4
\end{aligned}
\tag{2.4}
$$

where $P_i(i = 0, \cdots, 4)$ are defined by the control points $C_{i,j}$ $(j = 0, 1, 2)$ of the quadratic Bézier curve segments

$$
\begin{aligned}
P_0 &= C_{i,0} \\
P_1 &= C_{i,1} \\
P_2 &= C_{i,2} = C_{i+1,0} = (1-\lambda)C_{i,1} + \lambda C_{i+1,1} \\
P_3 &= C_{i+1,1} \\
P_4 &= C_{i+1,2}
\end{aligned}
\tag{2.5}
$$

For a given γ in order to guarantee G^2 continuity at the ends, the following constraints are derived:

$$a = \sqrt{\frac{3\gamma}{2\lambda}} \tag{2.6}$$

$$b = 1 - \sqrt{\frac{3(1-\gamma)}{2(1-\lambda)}} \tag{2.7}$$

We can adopt λ for γ as an initial value. Fig. 4 shows a quartic Bézier curve with $\gamma = \lambda$. Notice that the shape is approximated well by the quartic curve, but the positions of the curvature extrema are not preserved.

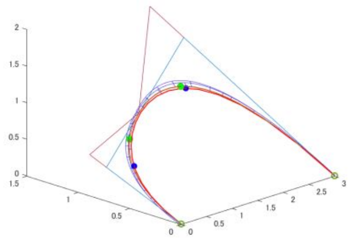


Fig. 4: Whole Replacement with a quartic Bézier curve.

Partial replacement with Quartic Bezier Curves:

Here we describe a method to replace the joint part of two adjacent quadratic curve segments with a quartic Bézier curve. We replace the parts of parameter intervals $\{t_1, 1\}$ and $\{0, t_2\}$ from the first and second segments, respectively with a quartic Bézier curve as shown in Fig. 5. The control points of these

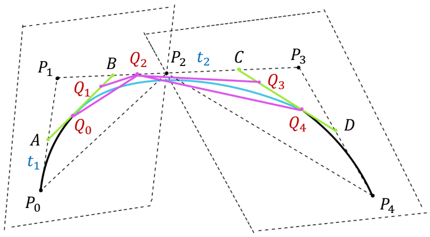


Fig. 5: Partial replacement control points setting.

parts are given by $\{A, B, C, D\}$ where

$$\begin{aligned} Q_0 &= (1 - t_1)A + t_1 B \\ Q_1 &= (1 - a')Q_0 + a'B \\ Q_2 &= (1 - \gamma')B + \gamma' C \\ Q_3 &= (1 - b')C + b'Q_4 \\ Q_4 &= (1 - t_2)C + t_2 D \end{aligned} \tag{2.8}$$

with

$$\begin{aligned} A &= (1 - t_1)P_0 + t_1 P_1 \\ B &= (1 - t_1)P_1 + t_1 P_2 \\ C &= (1 - t_2)P_2 + t_2 P_3 \\ D &= (1 - t_2)P_3 + t_2 P_4 \end{aligned} \tag{2.9}$$

and P_i $(i = 0, \cdots, 4)$ are also defined by Eq. (2.5). Therefore a' and b' are given by

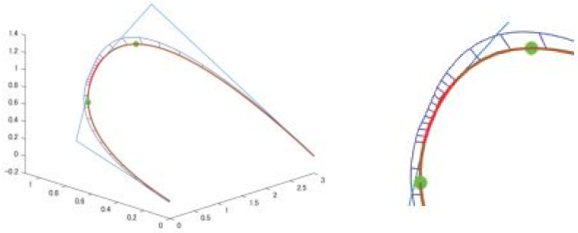
$$a' = \sqrt{\frac{3}{2}\frac{\gamma - \lambda t_1}{\lambda(1 - t_1)}} \tag{2.10}$$

$$b' = 1 - \sqrt{\frac{3}{2}\left\{1 + \frac{\lambda - \gamma}{(1 - \lambda)t_2}\right\}} \tag{2.11}$$

The concept of G^2 continuity as well as G^1 continuity is local and G^2 continuity is broken by the replacement with a quartic Bézier curve only at the joint of two consecutive quadratic Bézier curves. Hence we replace a partial segment at the joint of κ-space curve with a quartic curve to preserve the locations of its curvature extrema.

Results for κ-Space Curve:

The generated κ-space curve with the replacement of $C_i(t)$ for $t \in (0.9, 1)$ and $C_{i+1}(t)$ for $t \in (0, 0.1)$ is shown in Fig. 6. Notice that the curvature is continuous at the two joints with the quadratic curves. This replacement does not affect the locations of the original curvature extrema of κ-space curve in this case and we can make the replacement curve short as much as we like. Therefore we can preserve the locations of the curvature extrema.



(a) (b)

Fig. 6: Partial replacement with a quartic Bézier curve.

Conclusions:

This paper proposes a new method that enables to extend κ-curve to κ-space curve and maintain G^2 continuity by replacing quadratic Bezier curves with quartic Bezier curves.

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