Title:

Generalization of the Shape Uniqueness Theorem for Free-form Curves

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Introduction:

The shape uniqueness theorem for free-form curves shows the conditions on which the shapes of two parametric curves defined by three control points are identical altough their parametrization may be different [1]. According to this theorem, even though their blending functions look different, the curves become identical by reparametrizaion under some conditions on their blending functions. In this paper, we will extend this theorem for curves that are defined by four or more control points and show several examples of applications of the theorem.

Identical Shape of Free-from Curves:

Identical shape of two parametric curves is defined as follows [2]:

Definition 1. *For two parametric curves* $\boldsymbol{r} : I \to R^3$ *and* $\tilde{\boldsymbol{r}} : \tilde{I} \to R^3$*, there exists a* C^∞ *function* $\phi : I \to \tilde{I}$*,* 1) ϕ *is a one to one and onto mapping from* I *to* $\tilde{I}$*.* 2) ϕ *is strictly increasing.* 3) *For all* $t \in I$*,* $\tilde{\boldsymbol{r}}(\phi(t)) = \boldsymbol{r}(t)$*. We say that* $\boldsymbol{r}$ *and* $\tilde{\boldsymbol{r}}$ *define the same curve or their shapes are identical.*

Then $\tilde{\boldsymbol{r}}((\phi(t))$ is called reparametrization of $\boldsymbol{r}(t)$.

Uniqueness Theorem of the Shape of the Curve Defined by Three Control Points: [1]

In this paper, we assume that for $0 \le t \le 1$, a curve $\boldsymbol{C}(t)$ is defined by three control points $\boldsymbol{P}_0$, $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ as

$$\boldsymbol{C}(t) = u(t)\boldsymbol{P}_0 + v(t)\boldsymbol{P}_1 + w(t)\boldsymbol{P}_2 \tag{2.1}$$

where $0 \leq w(t) \leq 1$, $0 \leq v(t) \leq 1$ and

$$\begin{aligned} u(t)+v(t)+w(t) &= 1 \\ w(0) &= 0 \\ w(1) &= 1 \\ \frac{dw(t)}{dt} &> 0 \quad \text{for} \quad 0 < t < 1 \end{aligned} \tag{2.2}$$

We have removed the condition that $u(t) = w(1-t)$ from the original definition [1] since the theorem is still satisfied. If there is such a constant α that

$$v(t)^2 = \alpha u(t) w(t) \tag{2.3}$$

for $0 \leq t \leq 1$, then the following theorem is satisfied:

Theorem 1. *Uniqueness Theorem: The shape of the curve $\boldsymbol{C}(t)$ is determined by α exclusively and it does not depend on the basis functions $\{u(t), v(t), w(t)\}$ which are used to define the curve.*

Proof. For a given value $w_0 = w(t_0)$, $0 \leq w_0 \leq 1$, let $u_0 = u(t_0)$. Since $v(t) = 1 - u(t) - w(t)$,

$$(1 - u_0 - w_0)^2 = \alpha u_0 w_0 \tag{2.4}$$

Hence

$$u_0 = \frac{(\alpha-2)w_0 + 2 - \sqrt{\alpha w_0((\alpha-4)w_0+4)}}{2} \tag{2.5}$$

Since u_0 is uniquely determined by w_0, the location of the point $\boldsymbol{C}(t_0)$ is also uniquely determined because $\{u(t), v(t), w(t)\}$ are barycentric coordinates of triangle $\boldsymbol{P}_0\boldsymbol{P}_1\boldsymbol{P}_2$. By changing t from 0 to 1, $w(t)$ also increases from 0 to 1 and the shape of the curve $\boldsymbol{C}(t)$ is also completely determined. Q.E.D. ☐

Then $u(t) = u(w(t))$, $v(t) = v(w(t))$, and $w = w(t)$ are reparameterized blending functions. For example, the blending functions of quadratic Bézier curve $u(t) = (1-t)^2$, $v(t) = 2(1-t)t$, and $w(t) = t^2$ give $\alpha = 4$ and $u(w(t)) = (1 - \sqrt{w(t)})^2$, $v(w(t)) = 2(1 - \sqrt{w(t)})\sqrt{w(t)}$.

Figure 1 shows u_0 for $0 < w_0 < 1$ and $0 < \alpha < 10$.

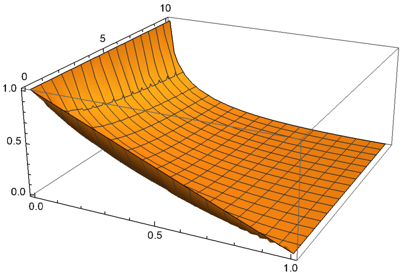

Fig. 1: u_0 for $0 < w_0 < 1$ and $0 < \alpha < 10$.

Generalization - The Case where Gobithaasan-Miura's Recursive Algorithm is satisfied:

In this section, we assume that the blending functions satisfy Gobithaasan-Miura's recursive algorithm [3]. Then

$$\begin{aligned}\boldsymbol{C}(t) &= u(u\boldsymbol{P}_0 + v\boldsymbol{P}_1 + w\boldsymbol{P}_2 + x\boldsymbol{P}_3)\\ &\quad + v(u\boldsymbol{P}_1 + v\boldsymbol{P}_2 + w\boldsymbol{P}_3 + x\boldsymbol{P}_4)\\ &\quad\quad + w(u\boldsymbol{P}_2 + v\boldsymbol{P}_3 + w\boldsymbol{P}_4 + x\boldsymbol{P}_5)\\ &\quad\quad\quad + x(u\boldsymbol{P}_3 + v\boldsymbol{P}_4 + w\boldsymbol{P}_5 + x\boldsymbol{P}_6)\\ &= u^2\boldsymbol{P}_0 + 2uv\boldsymbol{P}_1 + (2uw + v^2)\boldsymbol{P}_2 + 2(ux + vw)\boldsymbol{P}_3 + (2vx + w^2)\boldsymbol{P}_4 + 2wx\boldsymbol{P}_5 + x^2\boldsymbol{P}_6\end{aligned}$$

where the blending functions u, v, w, and x of parameter t are assumed to satisfy partition of unity. Hence for an arbitrary $t \in [0, 1]$,

$$u + v + w + x = 1 \tag{2.6}$$

is satisfied.

For the curve to be represented by seven control points with seven blending functions, the following equations must be satisfied:

$$\begin{aligned} v^2 &= \alpha uw & (2.7)\\ w^2 &= \beta vx & (2.8)\\ vw &= \gamma ux & (2.9)\end{aligned}$$

where $\alpha > 0$, $\beta > 0$, and $\gamma > 0$ are constants taht are independent from parameter t. However, the product of both sides of Eqs.(2.7) and (2.8) yields

$$\begin{aligned} v^2w^2 &= \alpha\beta uvwx \\ vw &= \alpha\beta ux & (2.10)\end{aligned}$$

Therefore

$$\gamma = \alpha\beta \tag{2.11}$$

When α and β satisfy Eqs.(2.7) and (2.8), respectively, Eq.(2.9) is automatically satisfied.

Therefore, if the blending functions u, v, w and x satisfy the following conditions, for a given function x the other functions u, v, and w are uniquely determined. Thus we can elevate the degree and increase the number of control points of the shape uniqueness theorem.

$$u + v + w + x = 1,\ v^2 = \alpha uw,\ w^2 = \beta vx. \tag{2.12}$$

The function $x(t)$ satisfies the followings:

$$\begin{aligned} x(0) &= 0,\\ x(1) &= 1,\\ \frac{dx(t)}{dt} &> 0. & (2.13)\end{aligned}$$

Theorem 2. *Shape Uniqueness Theorem of Higher Degree (#control points= 4): The shape of the curve* $\boldsymbol{C}(t)$ *is determined by* α *and* β *and it does not depend on the blending functions of use* $\{u(t), v(t), w(t), x(t)\}$.

Proof. For $x_0 = x(t_0)$ $(0 \leq x_0 \leq 1)$, we assume that $u_0 = u(t_0)$, $v_0 = v(t_0)$, and $w_0 = w(t_0)$. From Eqs. (2.7) and (2.8),

$$v_0 = \alpha^{\frac{2}{3}}\beta^{\frac{1}{3}}u_0^{\frac{2}{3}}x_0^{\frac{1}{3}}$$
$$w_0 = \alpha^{\frac{1}{3}}\beta^{\frac{2}{3}}u_0^{\frac{1}{3}}x_0^{\frac{2}{3}}$$

Since $u_0 + v_0 + w_0 + x_0 - 1 = 0$,

$$u_0 + \alpha^{\frac{2}{3}}\beta^{\frac{1}{3}}u_0^{\frac{2}{3}}x_0^{\frac{1}{3}} + \alpha^{\frac{1}{3}}\beta^{\frac{2}{3}}u_0^{\frac{1}{3}}x_0^{\frac{2}{3}} + x_0 - 1 = 0 \tag{2.14}$$

Let the left side of the above equation be $f(u_0; x_0)$. When $x_0 = 0$,

$$f(u_0; 0) = u_0 - 1 \tag{2.15}$$

Hence $u_0 = 1$. When $x_0 = 1$,

$$f(u_0; 1) = u_0^{\frac{1}{3}}(u_0^{\frac{2}{3}} + \alpha^{\frac{2}{3}}\beta^{\frac{1}{3}}u_0^{\frac{1}{3}}x_0^{\frac{1}{3}} + \alpha^{\frac{1}{3}}\beta^{\frac{2}{3}}) \tag{2.16}$$

Then $u_0 = 0$.

If we assume that $0 < x_0 < 1$,

$$f(0; x_0) = x_0^3 - 1 < 0$$
$$f(1; x_0) = \alpha^{\frac{2}{3}}\beta^{\frac{1}{3}}x_0^{\frac{1}{3}} + \alpha^{\frac{1}{3}}\beta^{\frac{2}{3}}x_0^{\frac{2}{3}} + x_0 > 0$$

Furtheremore

$$\frac{\partial f(u_0; x_0)}{\partial u_0} = 1 + \frac{2}{3}\alpha^{\frac{2}{3}}\beta^{\frac{1}{3}}x_0^{\frac{1}{3}}u_0^{-\frac{1}{3}} + \frac{1}{3}\alpha^{\frac{1}{3}}\beta^{\frac{2}{3}}x_0^{\frac{2}{3}}u_0^{-\frac{2}{3}} > 0 \tag{2.17}$$

Hence for x_0, $f(u_0; x_0)$ is a continuous function of u_0 and strictly increasing. Since $f(0; x_0) < 0$ and $f(1; x_0) > 0$, For x_0, u_0 is determined such that $0 \leq u_0 \leq 1$. $\{u(t), v(t), w(t), x(t)\}$ are barycentric coordinates of tetrahedron $\boldsymbol{P}_0\boldsymbol{P}_1\boldsymbol{P}_2\boldsymbol{P}_3$ and $\boldsymbol{C}(t_0)$ is uniquely determined. When t changes from 0 to 1, $x(t)$ changes 0 to 1 and the whole shape of the curve $\boldsymbol{C}(t)$ is determined completely. Q.E.D. $\square$

Note that even when tetrahedron $\boldsymbol{P}_0\boldsymbol{P}_1\boldsymbol{P}_2\boldsymbol{P}_3$ is degenerated into a 2D plane, the shape of the curve is uniquely determined by barycentric coordinates.

An application to the rational cubic Bézier curve:

It is well known that as a reparameterization of a rational Bézier curve of degree n, its weights w_i can be changed without changing the curve shape as follows: [4].

$$\hat{w}_i = c^i w_i; \quad i = 0, \cdots, n. \tag{2.18}$$

where $c \neq 0$ is a constant. For example, when $c = \sqrt[n]{w_0/w_n}$, then if we subdivide all weights by w_0, we obtain $w_0 = w_n = 1$. When $n = 3$,

$$u(t) = \frac{(1-t)^3 w_0}{f(t)},$$
$$v(t) = \frac{3(1-t)^2 t w_1}{f(t)},$$
$$w(t) = \frac{3(1-t)t^2 w_2}{f(t)},$$
$$x(t) = \frac{t^3 w_3}{f(t)}. \tag{2.19}$$

where $f(t) = (1-t)^3 w_0 + 3(1-t)^2 t w_1 + 3(1-t)t^2 w_2 + t^3 w_3$. On these blending functions,

$$\alpha = \frac{v(t)^2}{u(t)v(t)} = \frac{3w_1^2}{w_0 w_2}$$
$$\beta = \frac{w(t)^2}{v(t)x(t)} = \frac{3w_2^2}{w_1 w_3}$$

When $c = \sqrt[3]{w_0/w_3}$, $\hat{w}_0 = w_0$, $\hat{w}_1 = c w_1$, $\hat{w}_2 = c^2 w_2$, and $\hat{w}_3 = c^3 w_3$. Then

$$\frac{3\hat{w}_1^2}{\hat{w}_0 \hat{w}_2} = \frac{3w_1^2}{w_0 w_2}$$
$$\frac{3\hat{w}_2^2}{\hat{w}_1 \hat{w}_3} = \frac{3w_2^2}{w_1 w_3}$$

are satisfied. Therefore from the shape uniqueness theorem of higher degree (thr number of control points = 4), we know the shape is unchanged. Note that when thr number of control points = 3, the similar argument is satisfied. When $w_0 = w_3 = 1$ as "normalized', we obtain

$$\alpha = \frac{3w_1^2}{w_2},$$
$$\beta = \frac{3w_2^2}{w_1} \tag{2.20}$$

Conclusions:

In this study, we consider the case where the blending functions satisfy Gobithaasan-Miura's recursive algorithm [3]. A higher-order (third-order) version of the shape uniqueness theorem is presented. In the future, further improvement of the theorem to include more various cases can be study.

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