Title:
Cable-Knot Sculptures

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Introduction:

Mathematical knots can lead to intriguing geometrical sculptures. Figure 1 shows some results from an undergraduate study project called "The Beauty of Knots" [3]. Here, some simple mathematical knots have been laid out as symmetrical 3D space curves, using cubic B-splines. This typically required the symmetrical placement in 3D space of between 10 and 20 control points. Along these curves, a crescent-shaped cross-section has been swept. The modeling was performed in Berkeley SLIDE [4] or in JIPCAD [2,1], two home-brewed CAD modeling tools. Their design files look much like those of any geometrical CAD system; but their usefulness and appeal arise from their built-in procedural generators. In particular, they comprise powerful sweep generators, where an arbitrary cross-section can be swept along a 3D space curve, allowing much control of its size, orientation, and torsional twisting, – even allowing some gradual change of the basic shape of the cross-section. The resulting boundary-representation is then output as an .STL file to produce aesthetically pleasing sculptural maquettes, using layered fabrication on inexpensive 3D printers (Fig. 1).

Fig. 1: "The Beauty of Knots": (a) Knot 6_1. (b) Knot 5_2. (c) Knot 7_7.

Once a pleasing symmetrical knot layout has been found, the complexity of the knot can readily be increased by running a "multi-strand cable" (Fig. 2a) along the knot curve. The cable is twisting in such a way that different strands connect into other strands in a cyclical manner (Fig. 2b) and thus form again a single closed loop representing a much more complex mathematical knot. The basic approach is introduced by starting with an "Un-Knot" in the shape of a single circular loop (Fig. 3).

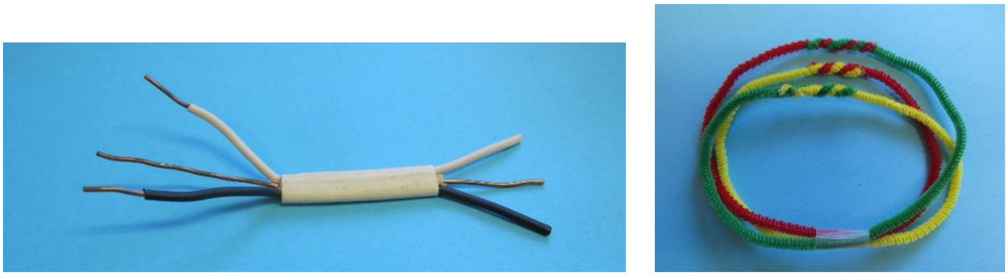

Fig. 2: (a) A 3-strand “Cable.” (b) Closing a twisted cable into a single-strand loop.

Unknots:
If we run a 3-strand cable along a simple circular un-knot and give the cable a 120-degree twist before joining its ends, we obtain a classical “1_3_Torus Knot” (Figs. 3a,b). Mathematically, this is still an un-knot, but now with a more interesting geometry.

We can further modify the geometry by warping the shape of the underlying donut around which the torus knot has been wound. Figure 3(c,d) shows two views of the modified geometry. The figures depict 3D-print models, about six inches tall.

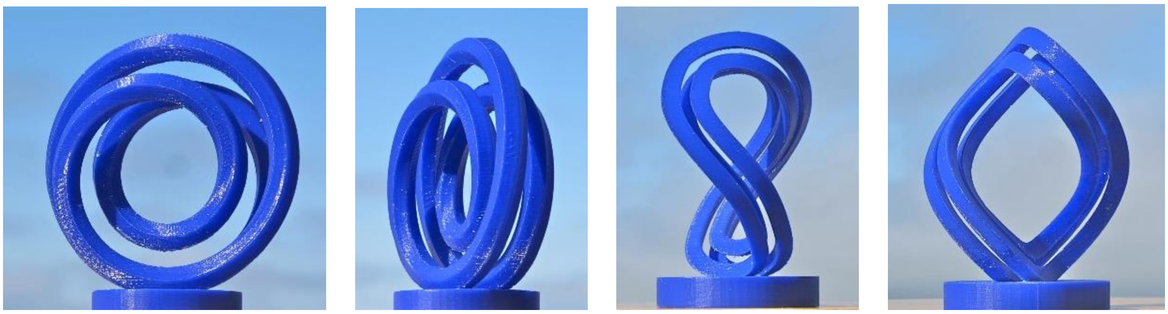

Fig. 3: Un-knots: (a,b) A simple torus knot, front and side. (c,d) Warped version of the same knot.

Trefoil Knots:
Things get more interesting when we let the cable itself form a real, non-trivial knot. The simplest such knot is the trefoil, – a 3-crossing knot, called knot_3_1. Figure 4a exhibits its classical layout with 3-fold rotational symmetry. If we run a 3-strand cable along this curve and give the cable a 120˚-twist before closing, we obtain a much more complicated knot (Figs. 4b,c). Wherever the simple trefoil exhibited a single crossing, we now find two groups of three strands each crossing over one another, thus generating 3×9 crossings. The 120˚-twist of the cable adds two more crossings, since one strand needs to cross over the two other ones. Thus, overall, we end up with a 29-crossing knot. Figure 4b shows a CAD model that exhibits the cyclical connectivity. Figure 4c depicts a 3D-print model.

To generate such 3-strand cable-knots, we could run a special sweep along the trefoil curve, using a compound cross-sectional profile that comprises three separate patches with the chosen cross-section for the strand, – in this case, an equilateral triangle (Fig. 4d, top). We then apply a 120˚-twist to the cable as a whole. But there is another way to achieve the same result. We can use a simpler cross-section profile with just a single, offset strand-profile (Fig. 4d, bottom) and run this profile three times around the basic trefoil curve with an overall twist of 3×120˚ = 360˚. The azimuth rotation of the cable has been chosen in such a way that two strand-lobes touch the cylindrical base of the sculpture.

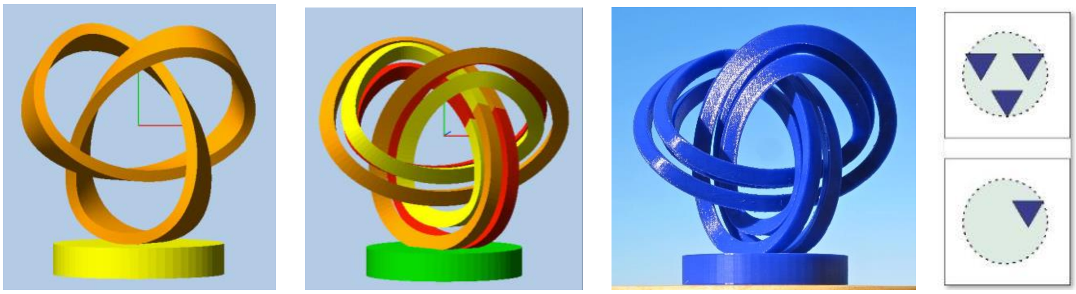

Fig. 4: (a) Trefoil knot. (b,c) two views of a 3-strand Trefoil cable-knot. (d) Cross-section profiles.

There is a different layout of the Trefoil knot that exhibits overall D_2 symmetry with three mutually orthogonal 2-fold rotation axes (Fig. 5a). Running a 3-strand, twisted cable along this curve results in the 2-lobe Trefoil cable-knot shown in Figures 5(b,c).

Fig. 5: (a) Trefoil knot with D_2 symmetry. (b,c) A “2-lobe” version of a Trefoil cable-knot, two views.

<u>Figure-8 Knots:</u>
The next more complicated knot after the trefoil is the 4-crossing Figure-8 knot. This knot has a pleasing, D_2-symmetrical layout (Fig. 6a) with four lobes, – two to stand on, and two more stretching towards the sky. Running a 3-strand cable along this knot curve, results in the Figure-8 cable-knot shown in Figures 6(b,c). Again, the azimuth rotation of the cable has been adjusted so that two strands in each of the bottom lobes touch the supporting base.

Fig. 6: (a) 4-lobe layout of the Figure-8 knot. (b,c) 3-strand Figure-8 cable-knot, two views.

Just as for the Trefoil knot, there exists a D_2-symmetric, 2-lobe layout of the Figure-8 knot (Fig. 7a). A corresponding 3-strand Figure-8 cable-knot is shown in Figures 7(b,c) from two different viewpoints.

Fig. 7: (a) 2-lobe layout of the Figure-8 knot. (b,c) A 2-lobe variant of the Figure-8 cable-knot.

<u>Further Enhancement of Knot Complexity:</u>
The complexity of these sculptural maquettes and of their underlying knots can be further enhanced in several different ways.

First, we can simply use cables with more strands. Figure 8a shows a 4-strand Figure-8 cable-knot that follows a knot curve that is more typical of what one would find in a mathematical knot table.

Alternatively, we could stick with a 3-strand cable, but let the three strands be more heavily intertwined by giving the cable a significantly higher amount of twist, – but still making sure that the three strands connect to one another in a cyclical manner. Figure 8b shows the result for a 3-strand cable-Trefoil with about ten full turns of cable twist.

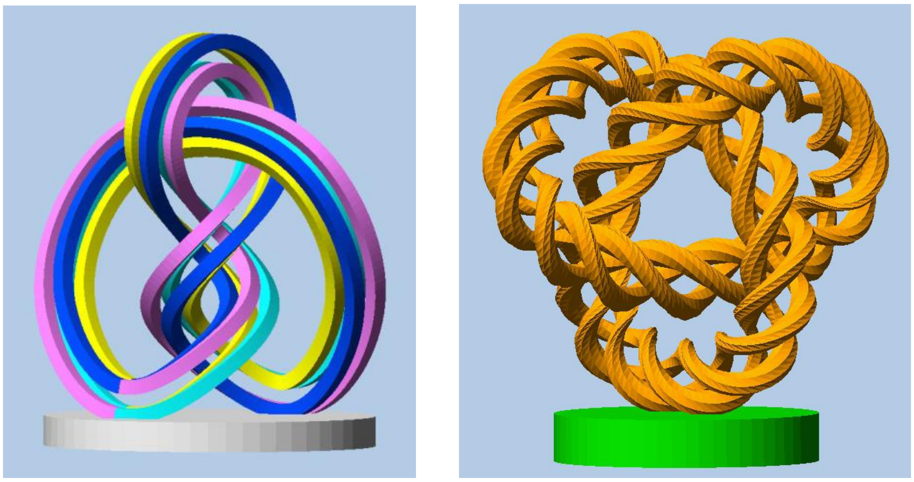

Fig. 8: (a) 4-strand Figure-8cable-knot with 90˚ twist. (b) 3-strand trefoil cable-knot with 3480˚ twist.

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One further step to increase knot complexity is to let the originally “parallel” strands be intertwined into a braid. Figure 9(a) shows a classical 3-strand braid. If we make sure that the whole braided loop is composed of a number of units that is <u>not</u> divisible by 3, then we obtain a cyclical connection of the three strands, resulting in a single-strand loop and thus in a single valid mathematical knot. This approach can lead to mathematical knots with hundreds of crossings. In Figure 9b such a 3-strand braid has been applied to the 2-lobe Trefoil knot (Fig. 5). Of course, more complicated multi-strand braids can be used (Fig. 9c). Figure 9d shows a loose, 5-strand braid applied to a simple un-knot.

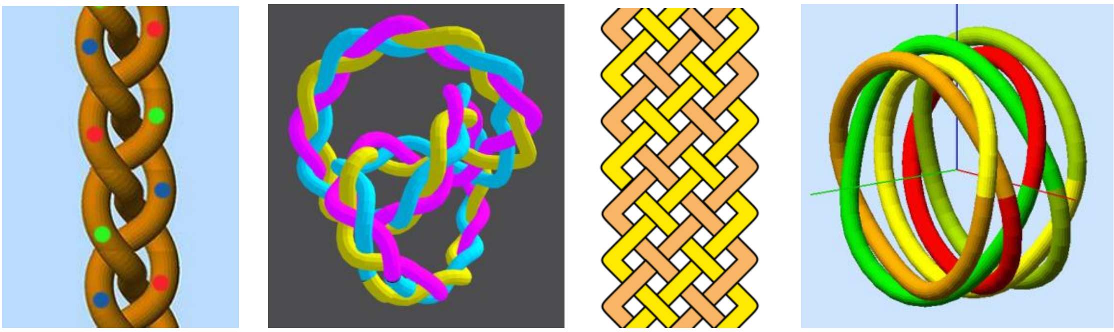

Fig. 9: (a) A classical 3-strand braiding pattern. (b) Trefoil knot with a tight 3-strand braiding. (c) A more complex 6-strand braiding pattern. (d) Un-knot with a loose, 5-strand braiding.

<u>Conclusion and Future Work:</u>

Cable-knots and their highly twisted or braided variants offer a rich playground for geometrical sculptures. First there are several topological parameters to choose from: the type of knot selected to place the cable, the number of strands, and the amount of twist when closing the cable loop.

In this study the focus has been on the basic topology and overall structure of the resulting sculptural forms. For this purpose, in most examples the individual strands have a simple triangular cross section. When building a real sculpture, additional geometrical issues must be addressed: The shape of the strand and its azimuthal orientation need to be carefully selected. In Figures 4(b,c) the triangular strand has been oriented so that one of its sides lies flat on the cylindrical pedestal. On the other hand, in Figures 6(b,c) the triangular strand penetrates into the base with one of its corners.

Furthermore, the cross-sectional profile need not remain constant along the whole sweep. As illustrated in Figures 7(b,c), the profile has been reduced in those parts of the sweep that go through the tightly linked middle of the sculpture and has been increased in the outer parts of the lobes.

Combining such geometrical fine-tuning with the almost limitless possibilities of starting with a cable-knot with enhanced complexity, resulting from heavily twisting or even braiding the cable strands, gives sculpture designers an inexhaustible set of possibilities.

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