Title:
Beta-Bezier Curves

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Introduction:
For years people have been trying to find ways to extend/modify the definition of Bezier curves so that one can change the shape of the curve without changing the control points of the curve [5-9]. But none of the works seem to be intuitive enough for practical applications in the field. A recent work by Chu and Zeng [10] was an attempt in that direction as well. A Beta-Bezier curve of degree n with shape parameter $\lambda \geq 0$ is defined by

$$\mathrm{C}(t;\lambda) = \sum_{k=0}^{n} P_k \beta_k^n(t;\lambda), \qquad 0 \leq t \leq 1$$

where

$$\beta_k^n(t;\lambda) = \binom{n}{k} \frac{\prod_{i=0}^{k-1}(\lambda t+i) \prod_{j=0}^{n-1-k}[\lambda(1-t)+j]}{\prod_{m=0}^{n-1}(\lambda+m)}, \tag{1}$$

are the so-called Beta-Bernstein basis functions, P_k are 2D or 3D control points.
When λ tends to infinity, Beta-Bezier basis functions reduce to Bernstein basis functions

$$B_{n,k}(t) = \binom{n}{k}(1-t)^{n-k} t^k$$

Hence the Beta-Bezier curves defined by Chu and Zeng [10] include Bezier curves as a special case. A few geometric properties of the curve, including a de Casteljau like algorithm similar to Bezier curve's de Casteljau algorithm, are studied [10]. Conditions on C^1-continuity at the joint of two adjacent Beta-Bezier curve segments are discussed by Levent and Sahin [11]. Unfortunately, Chu/Zeng and Levent/Sahin did not realize that the definition of Beta-Bezier curves given in [10] is not the best possible for Beta-Bezier curves.

In this paper, we present a new definition for Beta-Bezier curves and show that, with this new definition, properties of Beta-Bezier curves can be easily studied and computed. For instance, we are not only able to modify the shape of a Beta-Bezier curve without changing the control points of the curve, but also to perform all the properties of a Bezier curve such as recursive subdivision, converting to a B-spline representation, joining two curve segments with C^2-smoothness and interpolating a set of data points with a composite cubic Beta-Bezier curves that is C^2-continuous. An important observation is that in the cubic case, a Beta-Bezier curve is actually also a Bezier curve.

The rest of the paper is arranged as follows. In section 2, a new definition of Beta-Bezier curves is presented and basic properties of Beta-Bezier curves are studied. More advanced properties of Beta-Bezier curves such as smooth (C^2-) joining of two curve segments, subdivision property, C^2-continuous interpolation and B-spline conversion are discussed in Sections 3, 4, 5 and 6, respectively. Concluding remarks are given in Section 7.

New Definition of Beta-Bezier Curves and Basic Properties:

A Beta-Bezier curve of degree n with shape parameter $\beta \geq 0$ is defined as follows

$$\mathrm{C}(t;\beta) = \sum_{k=0}^{n} P_k B_{n,k}(t;\beta), \qquad 0 \leq t \leq 1 \tag{2}$$

where $P_0, P_1, \ldots, P_n$ are 2D or 3D control points and

$$B_{n,k}(t;\,\beta) = \frac{\binom{n}{k}\prod_{i=0}^{n-1-k}(1-t+i\beta)\prod_{j=0}^{k-1}(t+j\beta)}{\prod_{m=0}^{n-1}(1+m\beta)}, \tag{3}$$

$k = 0, 1, \ldots, n$, are Beta-Bernstein basis functions of degree n. The basis functions defined in (3) are related to the basis functions defined in (1) in that $\beta = 1/\lambda$. We have the following immediate properties of Beta-Bézier curves:

(i) When $\beta = 0$, $C(t;\beta)$ reduces to a Bezier curve $C(t)$ of degree n defined as follows

$$C(t) = \sum_{k=0}^{n} P_k B_{n,k}(t), \qquad 0 \leq t \leq 1 \tag{4}$$

where

$$B_{n,k}(t) = \binom{n}{k}(1-t)^{n-k}t^k \tag{5}$$

are Bernstein basis functions of degree n.

(ii) A Beta-Bezier curve segment always starts at the first control point P_0 and ends at the last control point P_n.

(iii) The sum of the basis functions of a Beta-Bezier curve equals one for any t and β (the unit sum property). Hence Beta-Bezier curves also satisfy the "convex hull property".

(iv) A non-zero β applies a dripping force to the curve. The dripping force pulls the curve segment towards the base line segment $\overline{P_0P_n}$ of the curve. When $\beta = +\infty$, the curve coincides with the base line segment. See Figure 1 for the cases when $\beta = 0$, $\beta = 1$ and $\beta = +\infty$ for a cubic Beta-Bezier curve segment.

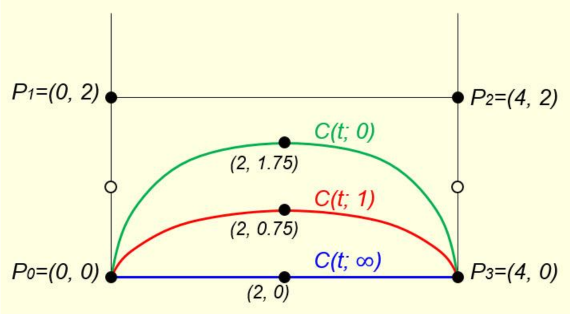

Fig. 1: Cubic Beta-Bezier curves defined by the same control points but with different shape parameters.

From this point on we will focus on degree 3 case only because that is what people are using for most of the applications.

(v) A cubic Beta-Bezier curve can be represented as a cubic Bezier curve.
Given a cubic Beta-Bezier curve $C(t;\beta)$ with control points P_0, P_1, P_2, P_3, defined as follows

$$C(t;\beta) = \sum_{k=0}^{3} B_{3,k}(t;\beta) P_k, \tag{6}$$

where $B_{3,k}(t;\beta)$ are Beta-Bernstein basis functions of degree 3 as defined in (3), through simple computation, one can rewrite $C(t;\beta)$ as a cubic Bezier curve as follows:

$$C(t;\beta) = (1-t)^3 P_0 + 3t(1-t)^2 Q_1 + 3t^2(1-t) Q_2 + t^3 P_3, \tag{7}$$

where

$$Q_1 = \frac{1}{1+\beta}\left(\frac{1+\beta}{1+2\beta} P_1 + \frac{\beta}{1+2\beta} P_2\right) + \frac{\beta}{1+\beta}\left(\frac{1+4\beta/3}{1+2\beta} P_0 + \frac{2\beta/3}{1+2\beta} P_3\right) \tag{8}$$

$$Q_2 = \frac{1}{1+\beta}\left(\frac{\beta}{1+2\beta} P_1 + \frac{1+\beta}{1+2\beta} P_2\right) + \frac{\beta}{1+\beta}\left(\frac{2\beta/3}{1+2\beta} P_0 + \frac{1+4\beta/3}{1+2\beta} P_3\right) \tag{9}$$

The computation process of Q_1 and Q_2 is shown in Figure 2.

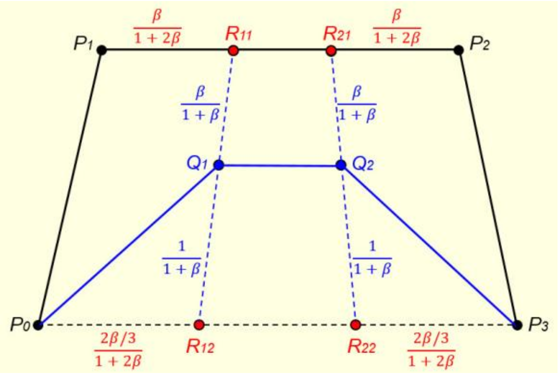

Fig. 2: Relationship between control points of a cubic Beta-Bezier curve and its Bezier control points.

(vi) Beta-Bezier curves have a de Casteljau-like algorithm [10].

When $\beta = 0$ the de Casteljau-like algorithm for Beta-Bezier curves reduces to the de Casteljau algorithm for Bezier curves. The de Casteljau algorithm for Beta-Bezier curves cannot be used in the recursive subdivision process of a Beta-Bezier curve though. Recursive subdivision technique for Beta-Bezier curves is shown in Section 4.

<u>Smoothness Conditions between Adjacent Curve Segments:</u>
Two cubic Beta-Bezier curve segments can be joined together with C^0-, C^1- or C^2-continuity. Details can be found in the full paper.

Recursive Subdivision:
Given a cubic Beta-Bezier curve segment $C(t;\ \beta)$ as defined in (6) and a $0 < t_0 < 1$, by computing the value of $C(t_0;\ \beta)$ the curve is divided into two Beta-Bezier curve segments at $C(t_0;\ \beta)$, each with its own control points. Therefore, one can perform recursive subdivision on cubic Beta-Bezier curves. An example illustrating the situation for a shape parameter $\beta = 1$ and $t_0 = 1/2$ is shown in Figure 3.

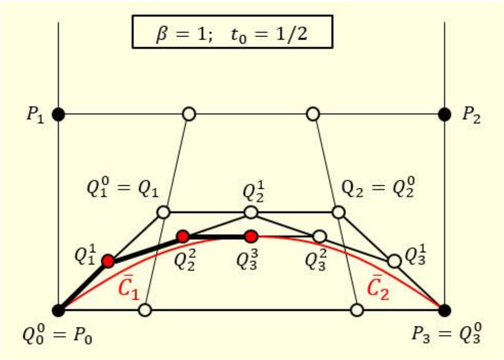

Fig. 3: Subdivision of a cubic Beta-Bezier curve segment with shape parameter $\beta = 1$ at $t_0 = 1/2$.

Interpolation using Composite Cubic Beta-Bezier Curves:
Given a set of data points D_0, D_1, D_2, $\cdots$, D_n, one can construct a composite cubic Beta-Bezier curve to interpolate these points. The curve is C^2- continuous. A closed curve example is shown in Figure 4. The interpolation process can be found in the full paper.

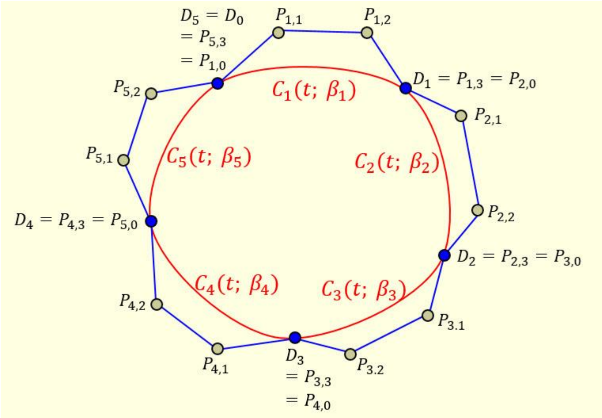

Fig. 4: Interpolation using a composite cubic Beta-Bezier curve.

Representation Conversion:
A cubic Beta-Bezier curve segment can be represented as a cubic B-spline curve segment.
Given a cubic Beta-Bezier curve $C(t;\ \beta)$ with control point set $\{P_0, P_1, P_2, P_3\}$ and shape parameter β, first convert it to a cubic Bezier curve as the one shown in (7) with Q_1 and Q_2 being defined as in (8) and (9). We then compute $\bar{P}_0$, $\bar{P}_1$, $\bar{P}_2$ and $\bar{P}_3$ as follows:

$$\bar{P}_1 = Q_1 + (Q_1 - Q_2); \quad \bar{P}_2 = Q_2 + (Q_2 - Q_1)$$

$$A_1 = P_0 + (P_0 - Q_1); \quad A_2 = P_3 + (P_3 - Q_2)$$
$$\bar{P}_0 = A_1 + 2(A_1 - \bar{P}_1); \quad \bar{P}_3 = A_2 + 2(A_2 - \bar{P}_2)$$

A cubic B-spline curve segment $CB(t)$ defined using $\bar{P}_0$, $\bar{P}_1$, $\bar{P}_2$ and $\bar{P}_3$ as its control points equals $C(t;\ \beta)$.

Concluding Remarks:

A new definition of Beta-Bezier curves is given. With the new definition, properties of Beta-Bezier curves are easier to study. It shows that Beta-Bezier curves not only have all the basic properties of Bezier curves, but also the capability of modifying the shape a Bezier curve segment or a C^2-continuous, composite cubic Bezier curve without changing the control points of the curve. This is because in the cubic case a Beta-Bezier curve is actually also a Bezier curve. Consequently, we have a curve design technique more general than Bezier curves. Since C^2-continuous, composite cubic Bezier curves are equivalent to uniform B-spline curves, this means the new curve design technique is more general than uniform B-spline curves as well.

Future works in this direction include the study of Beta-Bezier surfaces, extending the Beta shape parameter concept into B-spline curves and surfaces, and subdivision surfaces as well.

Acknowledgments:

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References:

[1] Farin, G.: Curves and Surfaces for Computer Aided Geometric Design – A Practical Guide. Academic Press, Cambridge, MA, USA. 1988

[2] Hoschek, J.; Lasser, D.: Computer Aided Geometric Design. A K Peters, Wellesley, MA, USA. 1993

[3] Gallier, J.: Curves and Surfaces in Geometric Modeling – Theory and Algorithms, Morgan Kaufmann, San Francisco, CA, USA. 2000

[4] Cheng, F.: Lecture Notes on Computer Aided Geometric Design, Chapter III. 2012, http://www.cs.uky.edu/~cheng/cs610/cs610-fall2012.html.

[5] Cao, J.; Wang, G. Z.: An extension of Bernstein-Bezier surface over the triangular domain, Progress Nat. Sci., 17, 2007, 352-357. https://doi.org/10.1080/10020070612331343269

[6] Han, X.-A.; Ma, Y.; Huang, X.: A novel generalization of Bezier curve and surface, Journal of Computational and Applied Mathematics, 217, 2008, 180-193. https://doi.org/10.1016/j.cam.2007.06.027

[7] Yang, L. Q.; Zeng, X.-M.: Bezier curves and surfaces with shape parameters, Int. J. Comput. Math., 86, 2009, 1253-1263. https://doi.org/10.1080/00207160701821715

[8] Yan, L. L.; Liang, J. F.: An extension of the Bezier model, Applied Mathematics and Computation, 218, 2011, 2863-2879. https://doi.org/10.1016/j.amc.2011.08.030

[9] Zhu, Y.; Han, X.: Quasi-Bernstein-Bezier polynomials over triangular domain with multiple shape parameters, Applied Mathematics and Computation, 250, 2015, 181-192. https://doi.org/10.1016/j.amc.2014.10.098

[10] Chu, L.; Zeng, X.-M.: Constructing curves and triangular patches by Beta functions, Journal of Computational and Applied Mathematics, 260, 2014, 191-200. https://doi.org/10.1016/j.cam.2013.09.025

[11] Levent, A.; Sahin, B.: Beta Bezier Curves, Applied and Computational Mathematics, 18(1), 2019, 79-94.