Title:
Calculation and Simulation of Elastic Deformation of Thin-walled Rings

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Introduction:
The ring is regarded as one of the most traditional and effective buffer structures because of its simple manufacture and low cost. It has been widely used in the collision energy dissipation system of aircraft, automobiles and other vehicles. And energy-absorbing devices composed of the ring row are also applied to the protection of large structures and highways. Early related research focused on the large plastic deformation of the ring, such as Deruntz et al. compressed the quasi-static low-carbon steel tube radially, and proposed that the ring was deformed into a four-hinge mechanism and its bearing capacity was calculated [2]; Yu Tongxi [3] explored the influence of plastic deformation on the bearing capacity of the ring by diameter stretching the ring; Sowerby et al. [4] first analyzed the influence of the point load on the radial direction of the ring on its deformation. However, when the ring collides at a lower speed, only elastic deformation occurs, and the current research on the elastic deformation mechanism of the ring is rare.

Studying the elastic deformation mechanism of the ring on the one hand makes the design of the existing ring buffer structure more reasonable by distinguishing between elastic and plastic deformation, and on the other hand, it provides a preliminary theoretical basis for the design of the new porous energy storage type buffer formed by the ring filling. Therefore, the paper does a detailed calculation and analysis of the elastic deformation of thin-walled rings. First, the deformation formula of the ring under quasi-static loading and the relationship between dynamic and static compression in free-fall condition are established, and the elastic coefficient expression of the ring is obtained. Then, the elastoplastic deformation analysis of the ring structure is carried out, and the expression of the elastic limit deformation amount is obtained. Finally, with the finite element method, the dynamic response law between parameter variables and ring performance under quasi-static loading and in free fall is studied and compared with the theoretical calculation results. The comparison results show that the simulation results are consistent with the calculated results, which validates the correctness of the derived deformation formula.

Main Idea:
Quasi-static Compression Deformation Calculation
The ring of radius R is deformed by a pair of concentrated forces P. Since the geometry and force of the ring are symmetrical to the two diameters perpendicular to each other, the deformation of the ring and the internal force are also symmetrical about the horizontal and vertical diameters, as shown in Fig. 1(a). From the equilibrium condition, the axial force $N_0=P/2$ and the shear force $Q_0=0$. The force method is used to solve the bending moment M_0. Taking the bending moment M_0 as the excess binding force, denoted as X_1, and obtain the basic static setting system as shown in Fig. 1(b). The corner of the

section is indicated by Δ_1, and the m-n section does not rotate, that is, Δ_1 is equal to zero. So

$$\Delta_1 = \delta_{11} X_1 + \Delta_{1P} = 0 \tag{1.1}$$

where Δ_{1P} is the rotation angle of the m-n section under the action of $P/2$, and δ_{11} is the rotation angle of the m-n section under the action of the unit force moment.

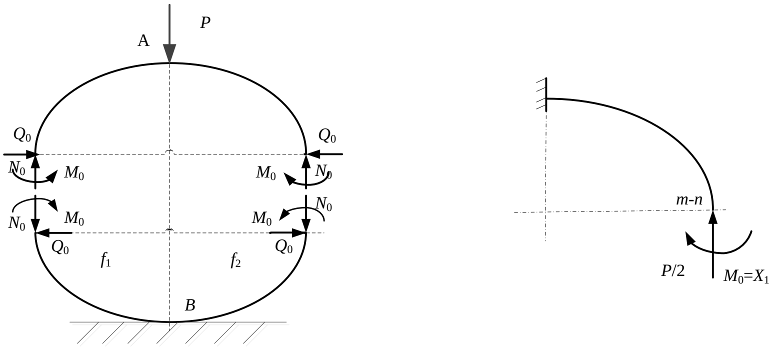

Fig.1: Ring force diagram: (a) Schematic diagram of internal, (b) Quarter ring force diagram.

Using the Mohr integral [5], find the values of Δ_{1P} and δ_{11} as:

$$\Delta_{1P} = \int_0^{\frac{\pi}{2}} \frac{M M_0 \mathrm{d}s}{EI_Z} = -\frac{PR^2}{2EI_Z}(\frac{\pi}{2}-1) \tag{1.2}$$

$$\delta_{11} = \int_0^{\frac{\pi}{2}} \frac{M_0 M_0 \mathrm{d}s}{EI_Z} = \frac{\pi R}{2EI_Z} \tag{1.3}$$

where s represents the arc length, E represents the modulus of elasticity and I_Z represents the moment of inertia of the curved section. Substituting Δ_{1P} and δ_{11} into Eqn. (1.1), and finding the value of X_1. Therefore, the bending moment of an arbitrary section of a ring under the action of $P/2$ and X_1 is:

$$M(\theta) = PR(\frac{1}{2} - \frac{1}{\pi}) - \frac{PR}{2}(1-\cos\theta) = \frac{PR}{2}(\cos\theta - \frac{2}{\pi}) \tag{1.4}$$

For round curved bars, if the central angle is expressed by θ, modifying the flexure differential equation of the ring [1] obtained by the card theorem according to the definition of the positive and negative directions of the above bending moment, and substituting the bending moment with Eqn. (1.4). Get the equation as

$$\frac{\mathrm{d}^2 w}{\mathrm{d}\theta^2} + w = \frac{PR^3}{2EI_Z}\cos\theta - \frac{PR^3}{\pi EI_Z} \tag{1.5}$$

where w is the radial displacement of the centerline (deflection) of the ring. Eqn. (1.5) is a second order constant coefficient linear differential equation. The radial displacement expression of any point on the ring subjected to the radial force P is obtained according to the boundary conditions:

$$w = \frac{PR^3}{4EI_Z}(\cos\theta + \theta\sin\theta - \frac{4}{\pi}) \tag{1.6}$$

When the cross section of the ring is a rectangle of length b and thickness t, the inertia $I_Z = bt^3/12$. According to the Eqn. (1.6), the elongation of the ring diameter CD is obtained as follows:

$$\delta_x \approx 1.664 \frac{PR^3}{Ebt^3} \tag{1.7}$$

The shortening of the ring diameter AB is:

$$\delta_y \approx 1.776\frac{PR^3}{Ebt^3} \tag{1.8}$$

The ring elasticity coefficient k is:

$$k = \frac{P}{\delta_y} = \frac{Eb}{1.776}\left(\frac{t}{R}\right)^3 \tag{1.9}$$

Calculation of Free-Falling Deformation and Calculation of Elastic Deformation Limit

The ring with a mass of m and an elastic coefficient of k falls free from a place where the vertical distance of the ring's bottom away from the rigid ground is h. Under the condition that no plastic deformation occurs, the ring will undergo slight deformation and then the deformation bounce will resume. It is assumed that the ring can compress its structure to the rest position after touching the ground, and the compression amount reaches the maximum, which is recorded as d_m. According to the law of conservation of energy and Hooke's law, the maximum compression d_m of the ring structure is obtained as:

$$d_m = \sqrt{\frac{2mgh}{k}} \tag{2.1}$$

When the material is statically compressed within the elastic limit range, its static deformation δ_s is:

$$\delta_s = \frac{mg}{k} \tag{2.2}$$

Therefore, d_m can be expressed as static deformation δ_s as:

$$d_m = \sqrt{2h\delta_s} \tag{2.3}$$

Some of the theories and calculations in this paper are based on linear elastic deformation, so the elastic deformation limit is needed. From the analysis of Eqn. (1.4), it can be seen that on the whole ring, when $\theta=\pi/2$ or $\theta=3\pi/2$, $|M(\theta)|$ is the largest. The elastic deformation range is $|M(\theta)|<M_e$, and M_e is the elastic limit bending moment [1]. Calculated to obtain:

$$P < \frac{\pi bt^2\sigma_s}{6R} \tag{2.4}$$

where b is the length of the ring, t represents the thickness, and σ_s is the yield strength of the material. Substituting Eqn. (2.4) into Eqn. (1.8), the elastic limit deformation of the ring is calculated that:

$$[d] = 0.93\frac{\sigma_s R^2}{Et} \tag{2.5}$$

From Eqn. (2.5), the elastic limit deformation of the ring is related to the yield strength and elastic modulus of the ring material and the outer radius and thickness of the ring. The greater the yield strength of the material, the greater the elastic limit deformation of the ring, so in the actual application, the material with better elasticity can be selected.

Simulation of Ring Deformation under the Action of Radial Concentration Force

Create a semi-circular plane model with thickness t=5mm and outer radius R=100mm. Assuming that the elastic modulus of the material is EX=2e3N/mm^2, Poisson-Pine ratio is $PRXY$=0.3. After applying the load, the post-processing module can display deformation and stress results.

In order to explore the relationship between the compression deformation of the ring and the ring parameters, change the parameter values for each simulation in the command flow, and get the Tab. 1. The first group shows the value of the ring compression δ_Y after the pressure P changes. The result of fitting with MATLAB is δ_Y=-1.445P-0.0005307, besides the SSE and constant terms are very close to zero. Therefore, δ_Y and P are in a linear function relationship. Select the 2nd set of data, the fitting result is δ_Y =-7.61/b, the SSE is 5.164e-24, close to zero, indicates δ_Y is inversely proportional to b. Fitting the δ_Y-t curve, shows that δ_Y =-693.3$t^{-2.974}$, So δ_Y and t are power function relationships, exponent is -2.974, close to -3. The fitting result of R is δ_Y =-5.931e-06*$R^{2.994}$, δ_Y and R are power function relationships same as t, but exponent is 2.994. Compared with the Eqn. (1.8), we can draw a conclusion that the amount of compression deformation δ_y is in a linear function relationship with pressure P and inversely proportional to *b*. The reality is very consistent with the theory, indicating that the theoretical calculation formula is correctly derived.

However, there is an error in the relationship between the outer radius *R*, the thickness *t* and δ_Y. Change the value of *R*/*t* for each simulation and compare the results with the values calculated by the Eqn. (1.7) and (1.8) to analyze the effect of *R*/*t* changes on errors. As shown in Tab. 2.

Group	*Serial number*	*Outer radius R*	*Thickness t*	*Length b*	*Force P*	*Compression deformation* δ_Y
1	1	100	5.0	10	4	-5.779
	2	100	5.0	10	8	-11.558
	3	100	5.0	10	20	-28.896
	4	100	5.0	10	40	-57.792
2	5	100	10.0	1	4	-7.610
	6	100	10.0	2	4	-3.805
	7	100	10.0	10	4	-0.761
3	8	100	5.0	10	4	-5.779
	9	100	2.5	10	4	-45.445
	10	100	2.0	10	4	-88.243
4	11	50	5.0	10	4	-0.724
	12	150	5.0	10	4	-19.450
	13	200	5.0	10	4	-46.023
	14	250	5.0	10	4	-89.779

Tab. 1: Ring compression deformation with different parameter values.

	Outer radius R	*Thickness t*	*Length b*	*Compression deformation* δ_Y	*Elongation deformation* δ_X	*Calculated value* δ_Y	*Calculated value* δ_X
1	50	5	10	-0.724	0.312	-0.710	0.329
2	60	5	12	-1.252	0.545	-1.228	0.568
3	75	5	15	-2.442	1.074	-2.398	1.110
4	100	5	20	-5.779	2.568	-5.683	2.630
5	125	5	25	-11.271	5.040	-11.100	5.138
6	250	5	50	-89.779	40.668	-88.800	41.100
7	300	5	60	-155.050	70.382	-153.450	71.021

Tab. 2: Simulation and calculation results of rings with different *R*/*t* values.

Calculate the relative error according to the results in Tab. 2. The results are represented by a polyline chart, as shown in Fig. 2. It can be clearly seen from the figure that as the values of radius and thickness gradually increase, the relative errors in both directions become smaller and smaller. That's because the theoretical calculation uses a thin-walled ring, which considers the ring as linear. Therefore, the larger the ratio of radius to thickness, the closer it is to linearity, and the closer it is to the theoretical calculation.

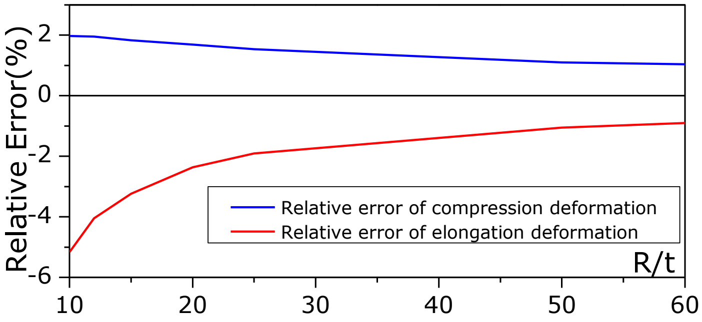

Fig. 2: Deformation amount relative error line chart.

Numerical Analysis and Elastoplastic Analysis of Free Fall Ring

Specify the analysis type as LY-Dynamic analysis, establish a cylindrical shell solid model with inner radius *r*=90mm, wall thickness *t*=10mm, length *b*=200mm, ρ=7.82e-9t/mm^3, *E*=21.1e4N/mm^2 and μ=0.288. Establish a desktop model 10mm from the bottom end of the cylinder, i.e. a rigid plane. The initial velocity of 2384.1141mm/s is applied to all nodes of the ring (which is the speed from the free fall of 290mm), and set the gravity acceleration to 9800mm/s^2. Set analysis time, calculate output parameters, energy consumption control, and then solve the model.

Entering the post-processor, generating a stress change animation of the thin-walled tube falling on the ground with time, the animation shows that the thin-walled ring has slightly deformed after contacting the ground, and then restores to the original shape. Use the capture animation command to get Fig. 3(a). The figure shows that the maximum displacement of the thin-walled tube is 11.1483mm. Because the initial distance is 10mm, the thin-walled tube has a compression deformation of 1.1483mm. After entering the time process, the processor generates a displacement diagram of a node, as shown in Fig. 3(b). The node drops in the first 0.00465 seconds, and then rebounds to the lowest point and then rebounds. The highest point after the rebound is less than the starting point, and the displacement change is at most -11.1467mm.

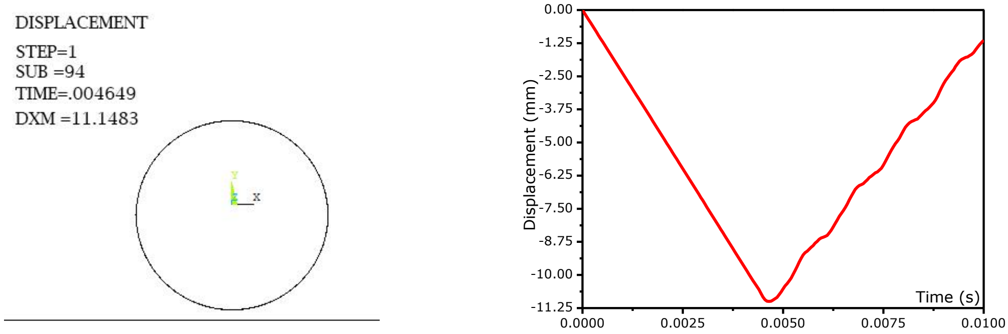

Fig. 3: Process result: (a) Displacement animation capture image, (b) Node displacement change.

In the above example, the thin-walled circular tube is deformed after landing, and the original shape is restored after the rebound, indicating that elastic deformation has occurred. When the drop height is changed to 1000m, the circular tube is obviously plastic deformation, theoretically deducing the height of the thin-walled circular tube for elastic limit deformation.

$$[d] = 0.93\frac{\sigma_s R^2}{Et} = 0.93\frac{430\times 100^2}{211\ 000\times 10} = 1.895\ 3\ \text{mm}$$

$$\delta = \frac{mg}{k} = \frac{[\rho\pi(R^2 - r^2)b]g}{\frac{Eb}{1.776}\left(t/R\right)^3} = \frac{9.336\times 10^{-3}\times 9\ 800}{23\ 761.26} = 0.003\ 85\ \text{mm}$$

Therefore:

$$h = \frac{d^2}{2\delta} \le \frac{[d]^2}{2\delta} = \frac{1.895\ 3^2}{2\times 0.003\ 85} = 466.5\ \text{mm}$$

That is to say, when the drop height *H*≤466.5mm, the thin-walled circular tube is elastically deformed, otherwise plastic deformation occurs.

Changing the height of the thin-walled tube from the ground, and obtain the maximum displacement of the thin-walled tube under different *H*. According to the data, the maximum compression amount will be plotted with height *H*, as shown in Fig. 4, where the blue color point is the data, and the red curve is the function curve fitted by the data in the elastic range. It can be seen from the figure that as the height increases, the difference between the simulated data value and the fitted curve becomes larger and larger, because it undergoes plastic deformation, which is different from the elastic deformation mechanism. However, when the height is 1000 mm, although it is greater than 466.5 mm, the simulation results are still not much different from the fitting results. The reason for the analysis is that only the elastic limit bending moment M_e is considered in the above calculation, and the plastic limit load M is not considered. In the range of M_e to M, although the outer fiber of the thin-walled circular tube has entered the plastic yielding stage, since the middle part is still in the elastic stage, the deformation characteristic of the "flat section" limits the plastic deformation of the outer layer fiber, so they are in the state of constrained plastic deformation. Therefore, when the height is 1000 mm, it can also conform to the elastic deformation curve.

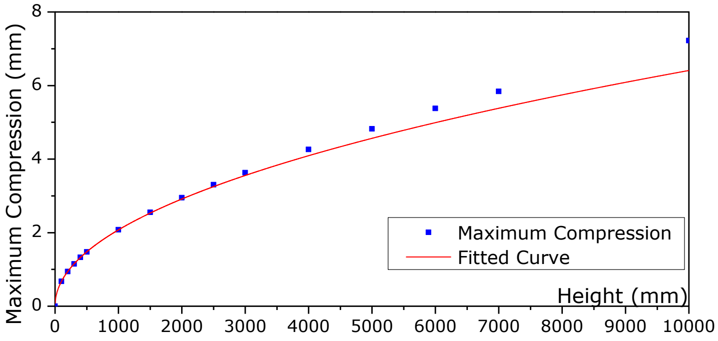

Fig. 4: Maximum compression at different heights.

Conclusions:

When the ring is elastically deformed under quasi-static compression, the relationship between compression deformation and the force is $\delta_y \approx 1.776\frac{PR^3}{Ebt^3}$, the elastic coefficient of the ring buffer cell is $k = \frac{Eb}{1.776}\left(\frac{t}{R}\right)^3$; under free falling , the relationship between the maximum of dynamic compression d_m and the static compression amount δ_s is $d_m = \sqrt{2h\delta_s}$. The elastoplastic deformation analysis of the ring buffer structure is obtained, and the elastic limit deformation $[d] = 0.93\frac{\sigma_s R^2}{Et}$. The dynamic response of the ring structure under quasi-static compression and free falling is simulated by ANSYS, and compared with the calculation results, the correctness of the above theoretical calculation formula is verified. The formula for calculating the elastic deformation of the ring provides a reference for the design of the energy storage buffer, which provides a theoretical basis for further research on the elastic deformation of the ring.

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