Title:
Regularized Set Operations in Solid Modeling Revisited

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Introduction:
The seminal works of Tilove and Requicha [8],[13],[14] first introduced the use of general topology concepts in solid modeling. However, their works are brute-force approach from mathematicians' perspective and are not easy to be comprehended by engineering trained personnels. This paper reviewed the point set topological approach using operational formulation. Essential topological concepts are described and visualized in easy to understand directed graphs. These facilitate subtle differentiation of key concepts to be recognized. In particular, some results are restated in a more mathematically correct version. Examples to use the prefix unary operators are demonstrated in solid modeling properties derivations and proofs as well as engineering applications.

Main Idea:
This paper is motivated by confusion in the use of mathematical terms like boundedness, boundary, interior, open set, exterior, complement, closed set, closure, regular set, etc. Hasse diagrams are drawn by mathematicians to study different topologies. Instead, directed graph using elementary operators of closure and complement are drawn to show their inter-relationship in the Euclidean three-dimensional space (E^3 is a special metric space formed from Cartesian product R^3 with Euclid's postulates [11] and inheriting all general topology properties.) In mathematics, regular, regular open and regular closed are similar but different concepts. The latter is the more correct term for Tilove and Requicha's works. Regular closed set also suffers the limitation that it can be bounded or unbounded. In solid modeling, valid solid is only a subset of compact (closed and bounded) set. However, no mathematical definition for valid solid can be found.

In order to maneuver properties in solid modeling, interior operator is more convenient. The operation approach is useful to derive the inclusion properties of dangling or pendant boundary and "open" boundary situations in traditional set intersection and difference respectively. Mathematically, arbitrary set in Euclidean three-dimensional space can be open, closed, clopen, and neither open nor closed, as well as with or without irregular boundary. Other than "open" and dangling (also called pendant) boundary, detached boundary and isolated points are also possible in general topology. As a result, regular closed set is verified to be necessary in constructive solid geometry representation. Many properties are also found to be more correctly stated as subset inclusion identities rather than equalities.

Constructive Solid Geometry (CSG):
Nowadays, with the aids of advanced computer-aided design (CAD) modelling tools, various applications such as advanced engineering product design and analysis, virtual reality (VR) and augmented reality (AR) visualizations or 3D printing of complicated physical components in different

disciplines are rapidly developed and implemented [5],[12]. In 3D model representation, solid modelling is normally employed and provide methods to overcome the limitations of wireframe and surface modelling [2],[7],[10]. The wire frame and surface modelling approaches have limited engineering applications due to insufficient of topological description and incompleteness in the geometric information. The use of solid modelling method can allow designers to create precise solid models with the aid of finite element analysis (FEA) under a simulate environment.

Accurate 3D complex models or assemblies can be designed and created by solid modelling. Besides, solid modelling can be employed to assess and evaluate the performances such as size, dimension, shape, functionality, or material utilization of complex products or assemblies during preliminary conceptual design stage.

CSG approach [1], [16] is one of the most popular solid representation methods with, user-friendly, accuracy, and validity. Nowadays, various CAD modeling tools uses CSG approach for product design and visualization, e.g. TinkerCAD [15]. A CSG model assumed that physical objected can be represented as a combination of simpler solid primitives. The primitives are cube, cylinder, cone, sphere, torus, etc. Instances of such primitive shapes are created. A complete solid model or assembly is created by combining these objects by Boolean Operations – union, difference, intersection. Boolean operations [9] are better modelling technique in 3D modelling systems that CSG allow users to create and modify models with convenient and easy editing capability among various representation schemes.

Mathematical Preliminaries:

This section highlights mathematical concepts essential to solid modeling. In 1922 Kuratowski reported in his breakthrough paper a maximum of fourteen distinct sets by alternate application of complement and closure to any set [3-4],[6]. The complement of arbitrary set A, denoted cA, is simply points does not belong to A. In other words, if U is the universal set, then cA=U-A. Closure can be defined in many ways. Relevant to solid modeling, the closure of A, denoted kA, is the disjoint union of the set of all isolated points of A and the set of all limit points of A. $p \in A$ is an isolated point of A, if and only if there exists an open neighborhood of p which does not contain any other points of A. A point $p \in U$ is a limit point (also called accumulation point or cluster point) of A if every open neighborhood of p contains another distinct point $q \neq p$ such that $q \in A$.

A point which has an open ball completely contained in A is an interior point of A. From these come three important definitions. The interior of a set A is the complement of the closure of the complement of A, iA=ckcA. The exterior of a set A is the complement of the closure of A, eA=ckA. The boundary (also called frontier) of a set A is the intersection of its closure and the closure of its complement, e.g. Eqn. (1):

$$\partial A = kA \cap kcA \tag{1}$$

Obviously, $\partial A = \partial cA$ can be expressed as below Eqn. (2):

$$\partial cA = kcA \cap kccA \tag{2}$$

To the layman, the boundary is considered as the set without its interior, i.e. A-iA. Confusion arises as shown in Eqn. (3) below is not equaled to ∂A but only its subset.

$$A - iA = A \cap ciA = A \cap kcA \tag{3}$$

Fig. 1 below shows an arbitrary set which is neither open nor closed, and with isolated points belonging to A and cA. Note that limit points can be interior points or boundary points. Isolated points are boundary points. Also, ∂A or ∂cA includes all boundary limit points and isolated points common to both A and cA.

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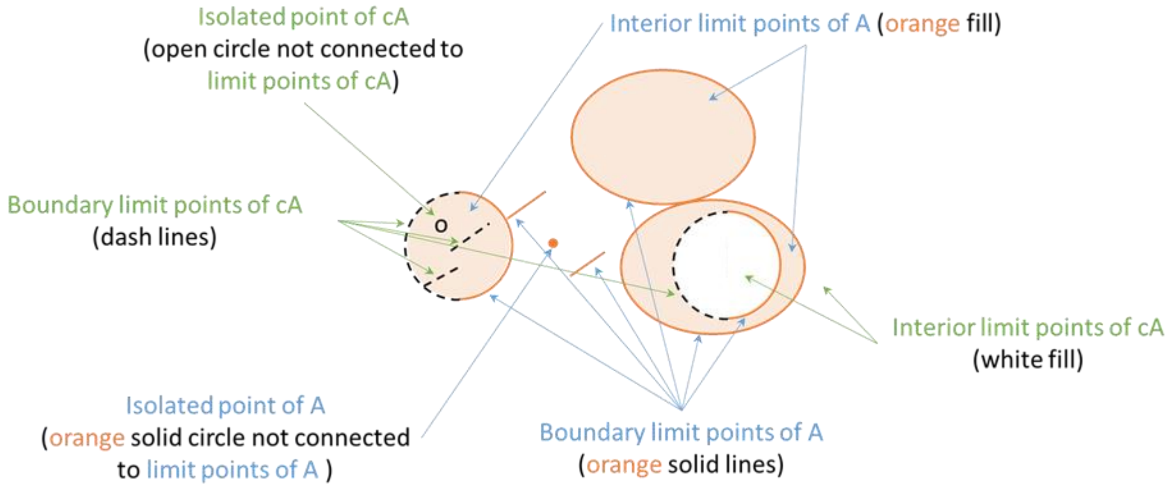
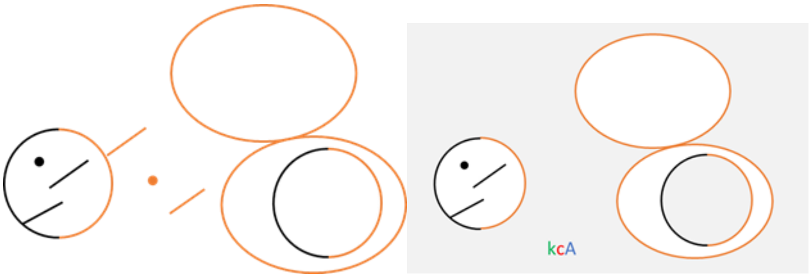


(a) isolated, limit, interior, and boundary points of A and cA



(b) ∂A=∂cA (c) kcA

Fig. 1: Differentiation of points of A and cA.

It should also be noted that kA includes boundary limit points of cA and irregular boundary points of cA are absorbed into limit points of A. Clarification is needed as sets can be open, closed, clopen (a portmanteau of closed-open), and neither open nor closed. A set is closed iff it contains all of its boundary points, i.e. A=kA. A set is open iff it contains all of its interior points, i.e. A=iA.

In topology, a clopen set in a topological space is a set which is both open and closed. In the Euclidean three-dimensional space of solid modeling, the only two clopen sets are the universal set U= E^3 and the empty set, e.g. Eqn. (4) and Eqn. (5):

$$kE^3=iE^3=E^3 \tag{4}$$

$$k\emptyset=i\emptyset=\emptyset. \tag{5}$$

Instead of Hasse diagram, the inter-relationships of Kuratowski's 14 sets are explicitly depicted as directed graphs. Fourteen distinct sets plus boundary created by closure and complement operators for arbitrary set are shown by Fig. 2. The legends are: prefix unary operators: c (complement), k (closure), i (interior), e (exterior), ∂ (boundary). Sets boxed in solid line are closed, in dashed line are open, and unboxed are undecided. Directed lines refer to mappings of respective operations.

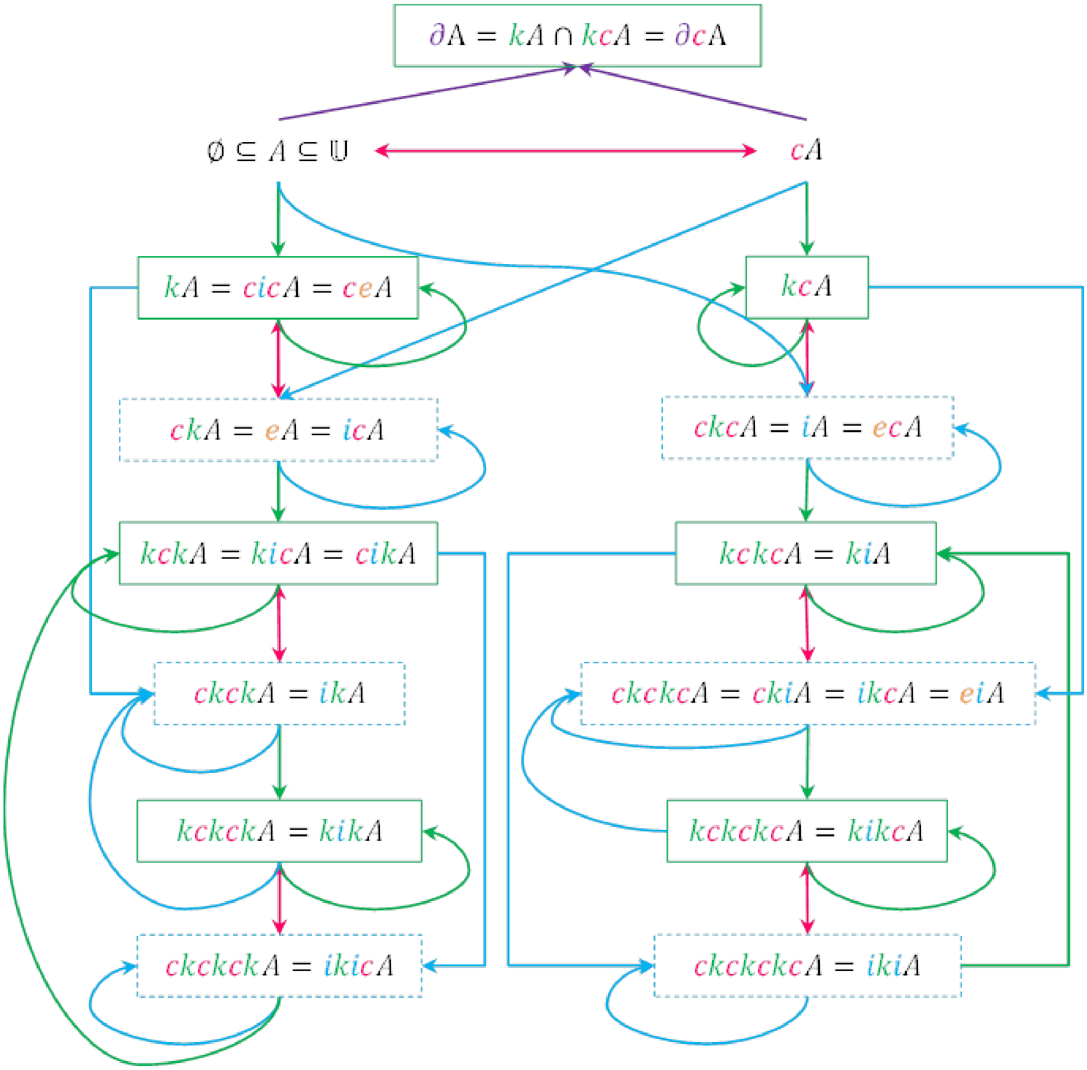


Fig. 2: Fourteen distinct sets plus boundary created by closure and complement operators for arbitrary set.

Conclusion:
This paper reviewed the point set topological approach using operational formulation and provided an alternate method to check the correctness of regularized set operations that was difficult to be previously formulated. An operational approach of general topology that may be more legible to CAD/CAM practitioners has been explained in detail. Some previous ad-hoc results are refined, and some new properties are derived. However, the above-mentioned operators are unable to detect non-manifold interim results in applying regularized set operations. Nevertheless, it is envisaged that the work presented will provide a solid foundation for future development. For instance, geometric modeling may be extended to using spatial reasoning via parallel operators to spatial modal logic. Essential topological concepts are described and visualized in easy to understand directed graphs with restated results in a more mathematically correct version. Examples to use the prefix unary operators

are demonstrated. Besides, the results simply provided the alternate approach that is easier to check the correctness for future research and development in geometric modeling.

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