Title:
Generalized Catenaries and Trig-Aesthetic Curves

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Introduction:

Beautiful curves have fascinated mankind since its beginnings, as seen from prehistoric images of circles and spirals, and the ornamental designs of antiquity. Drawing them required special tools, and draftsmen—especially in shipbuilding—used wooden or metallic *splines*. Fixing some points with weights, the elasticity of these strips ensured a generally smooth shape. Hoschek–Lasser [3] (Sec. 3.7) gives the following equations for these in terms of the curvature as a function of arc length:

$$\kappa''(s) = 0 \quad \text{(wooden)}, \qquad\qquad \int \kappa(s)^2\,\mathrm{d}s \to \min \quad \text{(mechanical)}. \tag{1}$$

The first equation means that curvature is a linear function, so it is a clothoid or Cornu spiral (also called an Euler spiral). The second leads to a differential equation studied thoroughly by (Jacob) Bernoulli and Euler, called the *elastica* [5].

In recent decades, the CAD community has tried to create curve representations that inherently possess some kind of *fairness*. One salient example is the log-aesthetic curve of Miura [6]. This is a generalization of several classic aesthetic curve representations, including the circle and its involute, the logarithmic spiral, the Cornu spiral and Nielsen's spiral.

In this paper we will look at a generalization and a trigonometric variant of this curve family. The rest of the paper is structured as follows. First we review the log-aesthetic curve formulation, then we apply various modifications to the equation and analyze its effects.

Preliminaries:

Log-aesthetic curves are based on the observation that the logarithmic curvature histogram (LCH) of fair curves tends to be a straight line [2]. From this, one can derive [8] the Cesàro equation

$$\kappa(s) = (c_0 s + c_1)^{-1/\alpha} \tag{2}$$

for some c_0, c_1 and α scalar values. In other words, a given power of the curvature, parameterized by arc length, should be linear.

We can also calculate the tangent angle by integrating the above formula by s:

$$\theta(s) = \frac{\alpha(c_0 s + c_1)^{(\alpha-1)/\alpha}}{(\alpha-1)c_0} + c_2. \tag{3}$$

With this, we can already plot the curve:

$$\mathbf{C}(s) = \mathbf{P}_0 + \left(\int_0^s \cos\theta(s)\,\mathrm{d}s, \int_0^s \sin\theta(s)\,\mathrm{d}s \right), \tag{4}$$

or even more compactly as $C(s) = P_0 + \int_0^s e^{i\theta(s)}\,\mathrm{d}s$ in the complex plane.

Note the special case of $\alpha = 0$ (Nielsen's spiral), where the equations for the curvature and the tangent become

$$\kappa(s) = \exp(c_0 s + c_1), \qquad \theta(s) = \frac{\exp(c_0 s + c_1)}{c_0} + c_2. \tag{5}$$

Not all of the parameters (α, c_0, c_1, c_2, $\mathbf{P}_0$) affect the shape. Miura [6] proves that this family of curves is self-affine, i.e., any 'tail' of a given curve can be affinely transformed to represent the whole, so we can discount any affine transformations introduced by the parameters. It is easy to see that $\mathbf{P}_0$ and c_2 control the starting position and tangent, respectively. Altering c_1 selects the starting s value (since $c_0 s + (c_1 + \Delta c_1) = c_0(s + \Delta c_1/c_0) + c_1 = c_0\hat{s} + c_1$), while c_0 and c_1 are jointly responsible for the scaling. Similarly to the 'standard form' of [8], let us set these parameters such that

$$\mathbf{C}(0) = \mathbf{0}, \qquad \theta(0) = 0, \qquad \kappa(0) = 1, \qquad \kappa'(0) = 1. \tag{6}$$

It is evident then that the α parameter—the slope of the LCH—is what defines the shape of the whole curve. The figure below shows some examples with different α values.

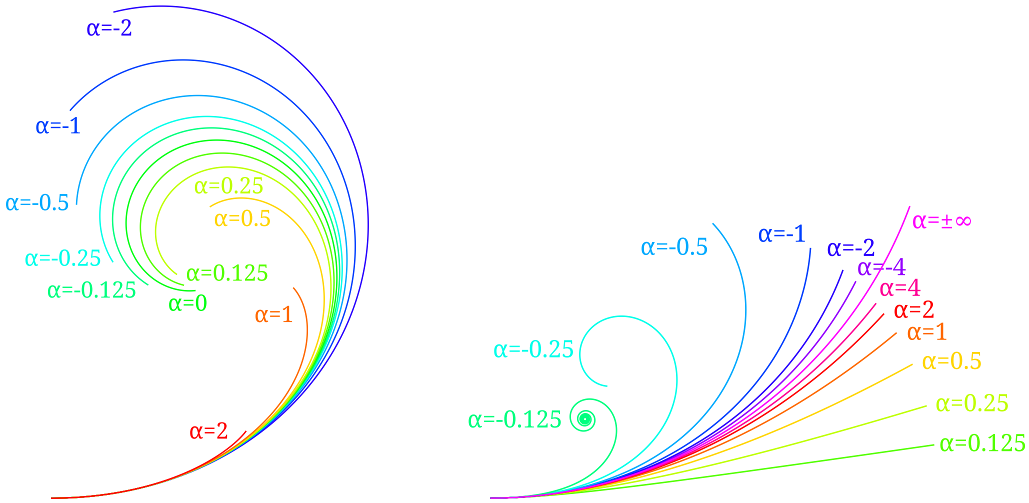

Fig. 1: Log-aesthetic curves with various α values. Left: in standard form (and thus different c_0, c_1, c_2 parameters). Right: with fixed $c_0 = c_1 = 1$ and $\theta(0) = 0$.

<u>Generalized Catenaries:</u>

We have seen that a natural requirement for aesthetic curves is that the curvature—or, as in log-aesthetic curves, a *power* of the curvature—should be a simple and smooth function, such as a straight line. A straightforward generalization would also allow parabolas:

$$\kappa(s) = (c_0 s^2 + c_1 s + c_2)^{-1/\alpha}. \tag{7}$$

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When $c_0 = 0$, we get back Eq. (2), and also when $c_1 = 2\sqrt{c_0 c_2}$ it can be converted to the form of Eq. (2):

$$\kappa(s) = (\sqrt{c_0}s + \sqrt{c_2})^{-\frac{1}{\alpha/2}}. \tag{8}$$

But other cases are significantly different, and this broader class of curves includes *catenaries*. The name comes from Latin *catena* (chain), as it describes the shape a hanging chain takes when supported at its endpoints. It has long been regarded as an aesthetic curve, and it is used in architecture for arches and domes. The definition is usually given as the graph of the function

$$y = a\cosh(x/a), \tag{9}$$

where a is a shape parameter. Its Cesàro equation is

$$\kappa(s) = \frac{a}{s^2 + a^2}, \tag{10}$$

which means $c_0 = \frac{1}{a}$, $c_1 = 0$, $c_2 = a$ and $\alpha = 1$ in Eq. (7).

The tangent angle for $\alpha = 1$ is

$$\theta(s) = D \cdot \arctan((c_0 s + \frac{1}{2}c_1)D) + c_3, \qquad \text{with } D = (c_0 c_2 - \frac{1}{4}c_1^2)^{-\frac{1}{2}}, \tag{11}$$

which simplifies to $\theta(s) = \arctan(s/a) + c_3$ in the case of catenaries. For general α values the equation for $\theta(s)$ gets fairly complex or even intractable. The case of $\alpha = -1$, however, presents no difficulties. An interesting subfamily is one where—similarly to catenaries—c_1 is set to 0. As before, set $\kappa(0) = 1$ and $\theta(0) = 0$, so the equation becomes

$$\kappa(s) = c \cdot s^2 + 1, \qquad \theta(s) = \frac{1}{3}c \cdot s^3 + s. \tag{12}$$

For positive c values, the curves resemble hyperbolic spirals, although their pitch angle is not quite right. (We will return to the topic of hyperbolic spirals and pitch angles in the next section.) For negative c values, they start off somewhat similarly to elastica (see Fig. 2 on the next page), but while the curvature of the spirals steadily increases, that of the elastica is periodic. With this in mind, we should turn next, in our quest for aesthetic curves, to periodic curvature functions.

Trig-Aesthetic Curves:

Cosine is probably the simplest periodic function we can use as curvature. We take Eq. (2) and replace exponentiation with cosine; and since we want to be able to express curvatures of any magnitude, we also multiply it by a constant, arriving at

$$\kappa(s) = c_0\cos(c_1 s + c_2), \qquad \theta(s) = \frac{c_0}{c_1}\sin(c_1 s + c_2) + c_3. \tag{13}$$

This turns out to be a curve used in geophysics (by the name *sine-generated curves*) to model *river meandering*, the wiggling shape a river takes as a result of erosion, transportation and deposition [4]. This natural process is actually connected with elastica, but the curve defined above is a good approximation.

The c_i constants have simple geometric meanings. Starting from the last one, c_3 controls the starting tangent, or in other words, rotates the curve; c_2 sets the starting parameter, so changing it shifts through the curve; and we can scale the curve by dividing both c_0 and c_1 by the scaling factor. The only parameter that really affects the *shape* is c_0, so in order to simplify our investigation, let us define *trig-aesthetic* curves as

$$\kappa(s) = \cos(s/c), \qquad \theta(s) = c\sin(s/c). \tag{14}$$

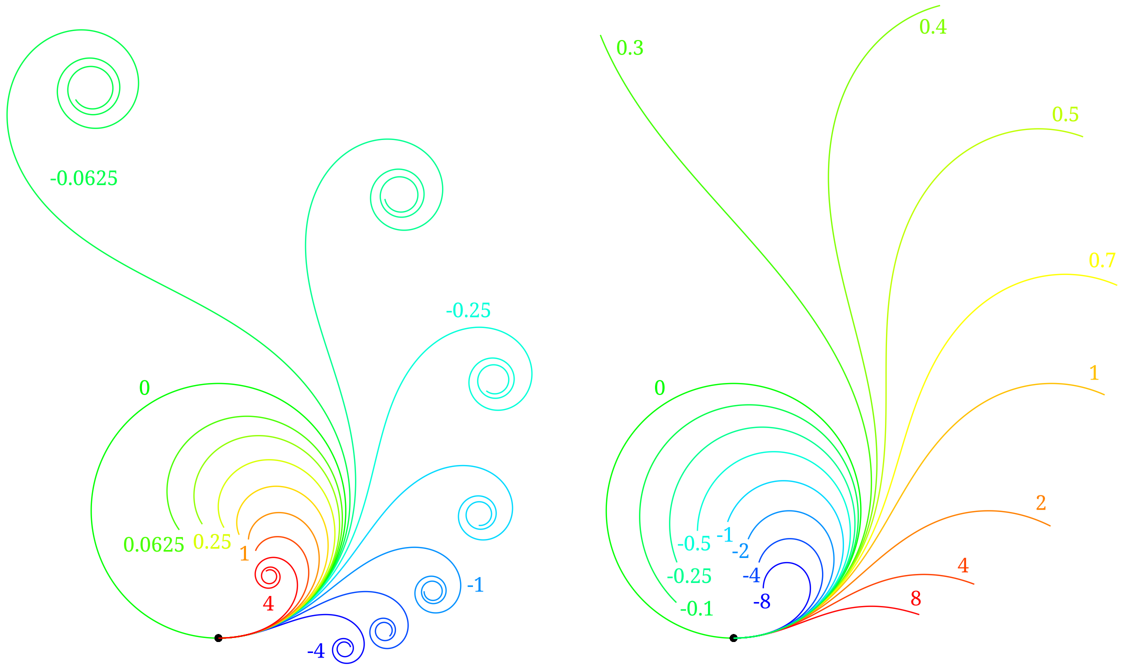

Fig. 2: Left: Curves with curvature $\kappa(s) = c \cdot s^2 + 1$. Values for c set as $\pm 2^k$, $k \in \{-4, -3, -2, -1, 0, 1, 2\}$, showing also the $c = 0$ circle. Right: *Elastica* – solutions to the differential equation $\theta''(s) + \lambda \sin\theta(s) = 0$ for various λ values.

This c parameter also has a geometric meaning: it is the maximum angle the curve can deviate from the starting tangent. (This angle is $\arccos(1 - \frac{1}{2\lambda})$ for the elastica, the amplitude of the related pendulum [7].)

The differential equation form of this curve family is

$$\theta''(s) + \theta(s)/c^2 = 0, \tag{15}$$

which differs only by a sign from that of the LA curves when $\alpha = 0$, $c_0 = 1/c$ and $c_2 = 0$:

$$\theta''_{\mathrm{LA}}(s) - \theta_{\mathrm{LA}}(s)/c^2 = 0. \tag{16}$$

If we also allow complex numbers for the c constant, using $c = -i$ results in

$$\kappa(s) = \cos(-s/i) = \cos(is) = \cosh(s), \qquad \theta(s) = \sinh(s). \tag{17}$$

This curve resembles the one in Eq. (12) with $c = \frac{1}{2}$, since $\cosh(s) \approx \frac{1}{2}s^2 + 1$ for small s values (easily seen from the Taylor expansion of $\cosh(s)$), and thus it is also similar to a hyperbolic spiral.

Spirals can be described by their *pitch angle*. At a given point of the spiral, draw a circle around the same center that touches that point. The pitch is the angle between the tangent of the spiral and the tangent of the circle. For example, logarithmic spirals have a constant pitch angle, while for Archimedean spirals it decreases as we get farther away from the center. In the case of hyperbolic spirals, the tangent of the pitch angle is proportional to the radius, i.e., the distance to the center. Numerical computation shows that for the curve above the tangent of the pitch angle converges to the radius, so the curve is indeed, in a sense, an approximation to the hyperbolic spiral.

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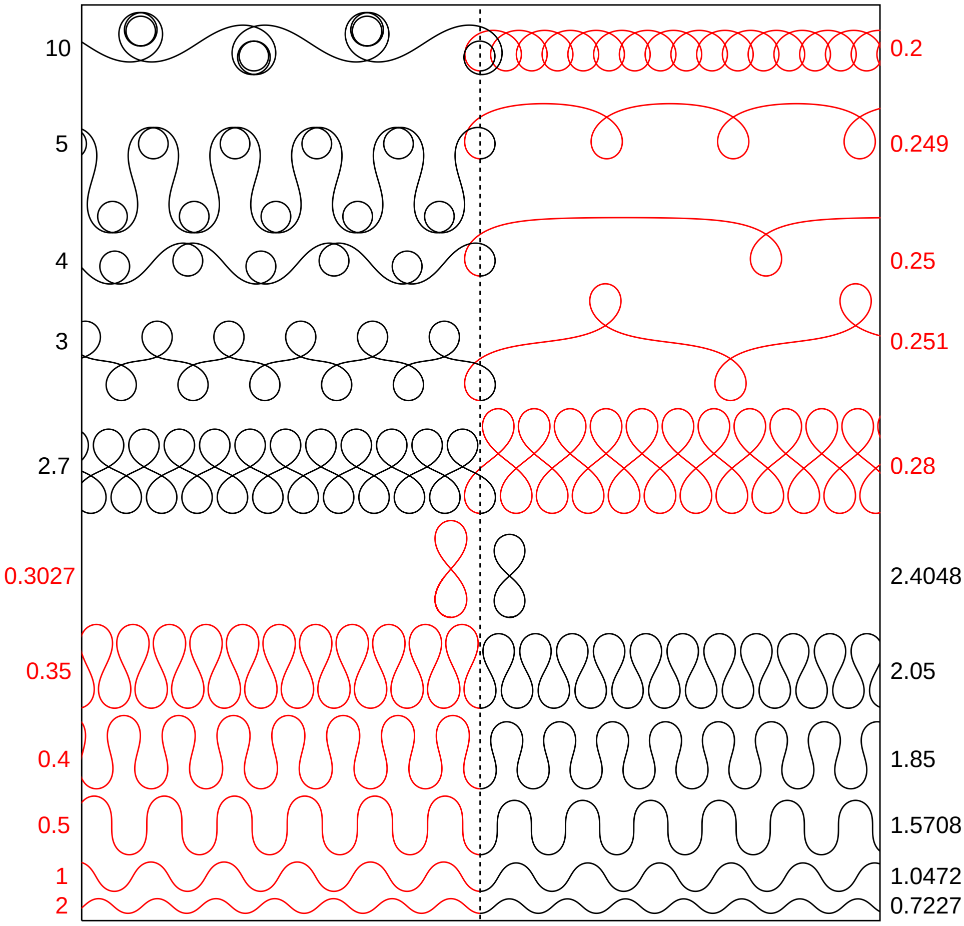

Fig. 3: Elastica (red) vs. trig-aesthetic curves (black) for various (λ and c) parameter values. For $\lambda > 0.25$ the corresponding c values were selected such that the curves should exhibit similar behavior (but not necessarily the same maximum angle). For $\lambda \leq 0.25$ there are no correspondences; the parameters are chosen to showcase the different shapes. This figure was inspired by Fig. 11 in [5].

The similarity between them can also be shown using the LCH slope (computed as $1-\rho(s)\rho''(s)/\rho'(s)^2$, see [1]), which is $-1-\cot^2(s/c)$ for trig-aesthetic curves, and thus $-1+\coth^2(s)$ when $c=-i$. This latter function quickly approaches 0, as does the slope of the hyperbolic spiral.

Conclusions:

A generalization of log-aesthetic curves has been given, which includes catenaries. We have also defined a variant curve family, which we called *trig-aesthetic* curves. The 'aesthetic' attribute is appropriate, because:

- It approximates (for a parameter range) the elastica family, which is an epitome of aesthetic curves.
- Its curvature is a simple, smooth function.
- It is very closely related to log-aesthetic curves, particularly to Nielsen's spiral.

- With complex parameters it approximates the hyperbolic spiral, another aesthetic curve.

The full paper will include more figures and detailed analysis, as well as an investigation of Hermite interpolation with trig-aesthetic curves.

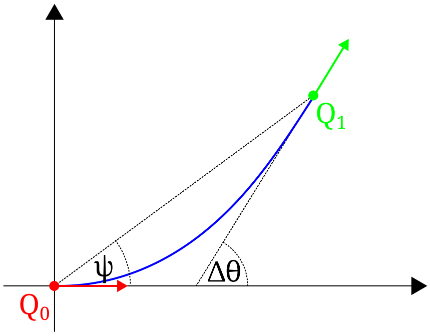

Fig. 4: Simplified Hermite interpolation problem (see the full paper for details).

It would be interesting to further generalize these curves to include or approximate other spirals in the Archimedean family (i.e., those with polar equation $r = a + b\phi^{1/n}$, such as the arithmetic spiral, the lituus or Fermat's spiral).

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