Title:

Interactive and Iterative Curvature Monotonicity Search and Region Visualization of Cubic Bézier Curve under Limited Hardware Resources

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Introduction:

In industrial manufacturing and product development, curve design is crucial because smooth and attractive designs require curves with uniform curvature variation, known as curvature monotonicity. In Computer-Aided Geometric Design (CAGD), Bézier curves are popular since their shapes can be easily adjusted by moving control points. However, manually choosing control points to maintain curvature monotonicity is both inefficient and challenging. This paper proposes a method to detect and visualize regions where potential control points ensuring curvature monotonicity exist. Although GPU-based visualization is powerful, not all users have access to high-performance GPUs. To overcome this limitation, the method uses a search grid and an iterative approach to evaluate and verify curvature monotonicity using the Curvature Monotonicity Evaluation Function (CMEF). This solution helps designers and engineers select appropriate control points more efficiently, leading to smoother curve designs and better products.

Research background:

Curvature monotonicity, relies on uniform curvature variation to ensure smoothness. The concept of Monotone Curvature Variation (MCV) was first rigorously defined by [1], who established both necessary and sufficient conditions for quadratic Bézier curves by adjusting the intermediate control point. Another study extended to conic segments as rational quadratic Bézier curves [2] and proposed conditions for achieving MCV in B-spline and planar Bézier curves to preserve desirable curvature while eliminating unwanted variations [3]. Sufficient geometric conditions have also been introduced by [4] to ensure monotonic curvature, leading to the construction of various polynomials with monotone curvature in Bézier curves, with further extensions to 3D class A Bézier curves [5]. More recent research [6], pointed out that while classical formulas determine curvature variation, evaluating monotonicity in higher-degree or rational curves requires complexity management, thus proposing a CMEF that uses any function yielding positive values when curvature increases and negative values when it decreases. Additionally, the real-time visualization of curvature monotonicity regions using GPUs has demonstrated promising methods for illustrating these regions, further enhanced by depicting them as intersections of half-spaces formed by implicit curves [7, 8].

Curvature Monotonicity Evaluation Function (CMEF) of Cubic Bézier curve:

This research implements the cubic Bézier curve to determine curvature monotonicity. The formulation of planar cubic Bézier, $\mathbf{Z}(u)$ is given by:

$$\mathbf{Z}(u) = \sum_{i=0}^{3} P_i f_i(u), \tag{2.1}$$

where P_i and f_i for $i = 0, 1, 2, 3$ denote the control points and the basis functions of the cubic Bézier curve respectively. The curvature monotonicity can be detected through the Curvature Monotonicity Evaluation Function (CMEF) as proposed by [6, 7, 8]. From Equation (2.1), $\mathbf{Z}(u)$ with degree n and $n+1$ control points, the curvature monotonicity can be determined by checking the numerator sign of $\frac{d\kappa}{ds}$. Then, the equation of $\frac{d\kappa}{ds}$ is denoted by

$$\frac{d\kappa}{ds} = \frac{(\dot{\mathbf{Z}} \times \dddot{\mathbf{Z}})(\dot{\mathbf{Z}} \times \dot{\mathbf{Z}}) - 3(\dot{\mathbf{Z}} \times \ddot{\mathbf{Z}})(\dot{\mathbf{Z}} \cdot \dddot{\mathbf{Z}})}{|\mathbf{Z}|^6}, \tag{2.2}$$

where $\dot{\mathbf{Z}}, \ddot{\mathbf{Z}}$, and $\dddot{\mathbf{Z}}$ denotes the first, second and third derivative of $\mathbf{Z}$ respectively. Furthermore, by assuming the denominator of Equation (2.2) is always positive, the numerator of the $\frac{d\kappa}{ds}$ can be expressed as Bernstein polynomial basis of degree $4n-7$ for planar curve which represented as $\lambda(u)$

$$\lambda(u) = \sum_{i=0}^{4n-7} B_i^{4n-7}(u)\xi_i, \tag{2.3}$$

where ξ_i are the corresponding coefficient and $B_i^{4n-7}(u)$ is the Bernstein basis. Next, based on the Theorem 1 and derivation of CMEF, $\lambda(u)$ from [6], the curvature monotonicity of polynomial planar curve with degree $n(n > 3)$ can be computed as

$$K_n = S_4(\mathbf{W}_1 \cdot \mathbf{W}_1) - 3S_3(\mathbf{W}_1 \cdot \mathbf{W}_2), \tag{2.4}$$

where

$$\begin{aligned}
\mathbf{W_1} &= n(\mathbf{R}_{11} - \mathbf{R}_{10}),\\
\mathbf{W_2} &= n(n-1)(\mathbf{R}_{22} - 2\mathbf{R}_{21} + \mathbf{R}_{20}),\\
S_3 &= n^2(n-1)(\mathbf{R}_{20} \times \mathbf{R}_{21} + \mathbf{R}_{22} \times \mathbf{R}_{20} + \mathbf{R}_{21} \times \mathbf{R}_{22}),\\
S_4 &= n^2(n-1)(n-2)((1-u)(\mathbf{R}_{31} - \mathbf{R}_{30}) \times (2\mathbf{R}_{31} - 3\mathbf{R}_{32} + \mathbf{R}_{33}) + u(\mathbf{R}_{30} - 3\mathbf{R}_{31} + 2\mathbf{R}_{32}) \times (\mathbf{R}_{33} - \mathbf{R}_{32})).
\end{aligned} \tag{2.5}$$

Note that, the ξ_i values are used to indicate whether the curvature monotonicity is increasing, decreasing or not monotonically varying. The indication of ξ_i values are as follows:

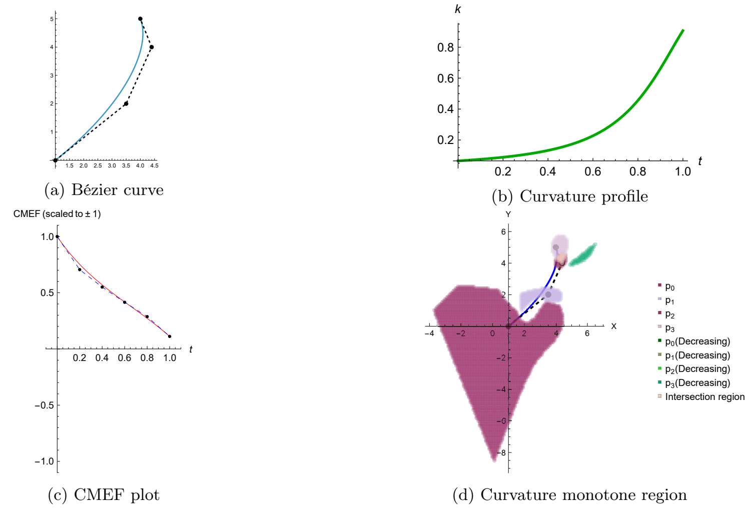
- If all the values of $\xi_i \geq 0$ for $i = 0, \ldots, 4n-7$, the curve is regarded as monotonically increasing.
- If all the values of $\xi_i \leq 0$ for $i = 0, \ldots, 4n-7$, the curve is regarded as monotonically decreasing.
- If $\xi_0 \cdot \xi_{4n-7} < 0$, the curve is regarded as non-monotonically varying.

Curvature Monotonicity Region:

In this section, the detection of the curvature monotonicity region is visualized and discussed. It should be noted that a few key components are required to verify the curvature monotonicity: the ξ_i values obtained from Equation (2.3), and the curvature profile. The ξ_i values can be visualized using the CMEF plot. Additionally, curvature monotonicity is characterized by a curvature profile that displays a consistently increasing or decreasing trend.

For clarity, the visualization of the curvature monotonicity region is based on two primary colours. Purple is used to indicate regions where the curvature increases, while green designates regions where the curvature decreases, with variations in the shades representing subtle differences. Note that, the curvature monotonicity region is determined for each individual control point. In other words, if control point is to be relocated to a different region, the adjustment must be performed one control point at a time. In addition to that, each control point of the curve has its own curvature monotonicity region and relocating them individually does not guarantee that the overall curvature will remain monotonic because these regions are based on the original configuration. Hence, once a control point is adjusted, either the code must be re-run to its initial setup or redo the iterative process is required to update the latest control points so that it will generate the latest curvature monotonicity region once the curvature profile does not monotone. Moreover, the search grid range applied for the following discussion is set from -10 to 10 along both x and y axis.

Furthermore, an example of curvature monotonic increasing profile is shown in Figure 1, where the Bézier curve exhibits a monotone increasing curvature profile as can be seen in Figure 1b. This observation further confirmed by Figure 1c, in which the plot is entirely contained within the positive quadrant. Figure 1d illustrates the region of the curvature monotonic increasing for each control point, while a curvature monotonic decreasing region is also identified for control point p_3. In other words, if curvature monotonic decreasing is desired, the control point p_3 may be relocated to the green region, although this adjustments does not guarantee the absence of inflection points. Moreover, an overlapping region between the curvature monotonic increasing area of p_2 and p_3 is represented by a peach colour.



(a) Bézier curve

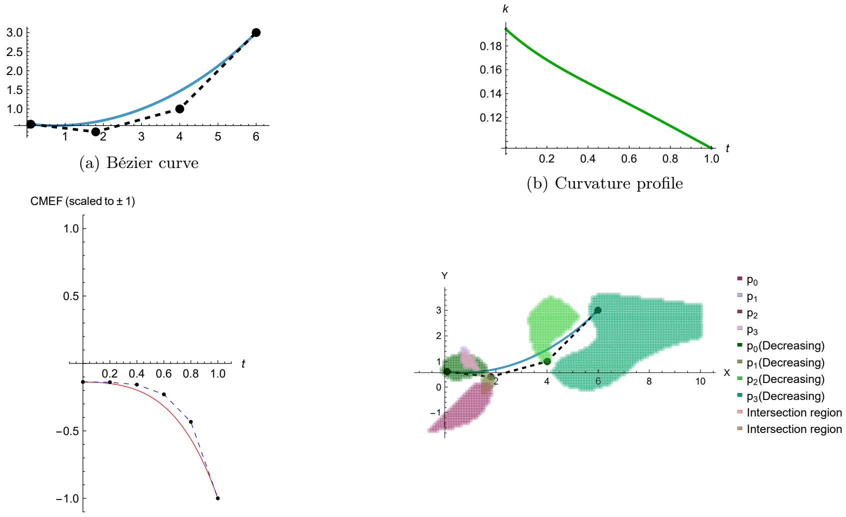
(b) Curvature profile

(c) CMEF plot

(d) Curvature monotone region

Fig. 1: Example of monotone curvature increasing.

Figure 2 demonstrates a case of curvature monotonic decreasing as depicted in Figure 2b. As portrayed in Figure 2c, the negative ξ_i values indicate a curvature profile that decreases monotonically. Next, the visualization of the curvature monotonicity region is shown in Figure 2d. The different shades of green represent the curvature monotonic decreasing region for each control point, while the purple regions indicate the curvature monotonic increasing region available for p_0 and p_1. Note that, the intersection region between the curvature monotone decreasing p_0 and curvature monotone increasing p_1 is coloured pink, whereas the intersection region of curvature monotone decreasing p_1 and curvature monotone increasing p_0 is coloured light brown.



(a) Bézier curve

(b) Curvature profile

(c) CMEF plot

(d) Curvature monotone region

Fig. 2: Example of monotone curvature decreasing.

Conclusions:

Based on the findings, this research uses an iterative grid search to identify control points that meet curvature monotonicity conditions. This approach provides designers with valuable insights to generate smooth curves efficiently. By visualizing the curvature monotonicity region, users can easily adjust the curvature according to their design requirements. Although it does not rely on a real-time GPU-based approach, it offers a practical alternative for those with limited resources or systems without advanced graphics capabilities. For simplicity, the study focuses on monotonicity regions without considering inflection points. One limitation is that each control point's monotonicity region is based on the original configuration, so modifying them individually does not ensure global monotonicity and requires iterative updates once all points are adjusted. Future research will aim to enhance this method by incorporating optimization techniques to automatically determine control point configurations that more efficiently satisfy curvature monotonicity.

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