Title:
Ratio Analysis of Blending Functions & Curve Conversion Using Shape Uniqueness Theorem

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Introduction:

In the ever-evolving fields of computer graphics and geometric modeling, the ability to classify, analyze, and transform free-form curves is a cornerstone of innovation, driving advancements across digital design, animation, and manufacturing. It has been a trend for researchers to modify Bezier curves for extra shape control using trigonometric functions and special functions, e.g.,Trigo-Bezier curves [2] and H-Bezier [3] & C-Bezier([4, 5]. Event though these curves have better control and can exactly represent of conic sections, they cannot be directly used in existing CAD/CAM systems due to thier special blending functions. The first part of this work explores the patterns of ratio property using Shape Uniqueness Theorem (SUT) [6] for Bezier, Rational Bezier and Ball curves. The second part leverages the SUT approach to establish a unified framework for curve conversion with constant ratios. To showcase its practicality, we implement the proposed method for quadratic C-Bezier conversion into Rational Bezier curves. The ratio-driven methodology bridges the gap between various curve types. It unlocks new possibilities for their transformation and application, reshaping how curves are understood and utilized in modern computational design.

Next section introduces the Shape Uniqueness Theorem (SUT) and its fundamentals, as established by Miura et al. [6]. Using SUT we establish the ratio properties of Bezier, Rational Bezier, and Ball [7] curves. It is followed by the conversion of quadratic C-Bezier curves to rational Bezier curves of degree two, elucidating the corresponding conic section formed based on chosen shape parameter value before concluding this work.

Shape Uniqueness Theorem (SUT)

Theorem 1. *The shape of the curve $C(t)$ defined by three control points P_0, P_1, P_2 is determined by the constant β exclusively and does not depend on the basis functions $u(t), v(t), w(t)$ used to define the curve. [6, 8]* □

Reader are referred to proof shown in [6]. Let $C(t)$ defined as follows for $0 \leq t \leq 1$:

$$C(t) = u(t)P_0 + v(t)P_1 + w(t)P_2 \tag{2.1}$$

where the following conditions hold:

1. $0 \leq w(t) \leq 1$
2. $0 \leq v(t) \leq 1$
3. $u(t) + v(t) + w(t) = 1$
4. $u(t) = w(1-t)$
5. $w(0) = 0$
6. $w(1) = 1$
7. $\frac{dw(t)}{dt} > 0 \quad \text{for} \quad 0 < t < 1$

If there exists a ratio β for $0 \leq t \leq 1$ such that

$$\beta = \frac{v(t)^2}{u(t)w(t)} \tag{2.2}$$

then the above theorem is satisfied. It is evident from SUT that its control points does not contribute to the ratio as shown in eqn (2.2).

Shape uniqueness theorem for Bezier curves:
Bezier curves are widely recognized for their precision and simplicity, making them a de facto standard for CAD/CAM systems. Surprisingly, the ratios of its blending functions are constant.

Definition 1: The general formula for the **ratio property of Bezier curve** for $1 \leq i \leq n-1$ can be expressed as follows:

$$R_i^n = \frac{(B_{i,n})^2}{(B_{i-1,n}) \cdot (B_{i+1,n})} = \frac{(i+1).(n-i+1)}{i(n-i)} \tag{2.3}$$

Using (2.3), we can obtain ratio property for any Bezier degree. Table 1 shows the ratio values up to degree 9.

Table 1: Ratio property of Bezier curve for different degrees

Ratio	$\frac{(B_{1,n})^2}{B_{0,n}B_{2,n}}$	$\frac{(B_{2,n})^2}{B_{1,n}B_{3,n}}$	$\frac{(B_{3,n})^2}{B_{2,n}B_{4,n}}$	$\frac{(B_{4,n})^2}{B_{3,n}B_{5,n}}$	$\frac{(B_{5,n})^2}{B_{4,n}B_{6,n}}$	$\frac{(B_{6,n})^2}{B_{5,n}B_{7,n}}$	$\frac{(B_{7,n})^2}{B_{6,n}B_{8,n}}$	$\frac{(B_{8,n})^2}{B_{7,n}B_{9,n}}$
Terms	R_1^n	R_2^n	R_3^n	R_4^n	R_5^n	R_6^n	R_7^n	R_8^n
$n=2$	4							
$n=3$	3/1	3/1						
$n=4$	8/3	9/4	8/3					
$n=5$	5/2	6/3	6/3	5/2				
$n=6$	12/5	15/8	16/9	15/8	12/5			
$n=7$	7/3	9/5	5/3	5/3	9/5	7/3		
$n=8$	16/7	7/4	8/5	25/16	8/5	7/4	16/7	
$n=9$	9/4	12/7	14/9	3/2	3/2	14/9	12/7	9/4
$\vdots$								

By deduction, the following ratio properties of Bezier's blengin function can be obtained:

1. Symmetry: For all degrees n, the sequence of ratios is symmetric $R_i^n = R_{n-i}^n$ for all $i \in \{1, \dots, n-1\}$.
2. First and last term: $R_1^n = R_{n-1}^n = \begin{cases} \frac{2k+1}{k}, & \text{if } n \text{ is odd, where } k = \frac{n-1}{2}, \\ \frac{4k}{2k-1}, & \text{if } n \text{ is even, where } k = \frac{n}{2}. \end{cases}$
3. Central terms: $\begin{cases} R^n_{\frac{n-1}{2}} = R^n_{\frac{n+1}{2}}. & \text{if } n \text{ is odd} \\ R^n_{\frac{n}{2}} \text{ is unique.} & \text{if } n \text{ is even.} \end{cases}$
4. Sequence length: The sequence $\{R_i^n\}$ contains exactly $n-1$ terms.

Shape uniqueness theorem for Ball curves:
Ball curve [7] known for its computational efficiency when it comes to degree reduction by coalescing control points. Specifically, a generalized Ball curve of odd degree $2m+1$ reduces its degree to 2m precisely when the centre two control points coincide ($P_m = P_{m+1}$). The Ball curve yields interesting results when analyzed through the lens of SUT. The ratios are not constants, but replies on its parameter t.

Definition 2: For $1 \le i \le n-1$, the general formula for the ratio property of Ball curve can be expressed as follows:

$$\textbf{Even degree, } R_i^n : \frac{(w_{i,n})^2}{(w_{i-1,n})(w_{i+1,n})} = \begin{cases} (1-t)^2, & \text{if } i = \frac{n-2}{2} \\ 4, & \text{if } i = \frac{n}{2} \\ t^2, & \text{if } i = \frac{n+2}{2} \\ 1, & \text{otherwise} \end{cases}$$

$$\textbf{Odd degree, } R_i^n : \frac{(w_{i,n})^2}{(w_{i-1,n})(w_{i+1,n})} = \begin{cases} 1-t, & \text{if } i = \frac{n-3}{2} \\ 2(1-t), & \text{if } i = \frac{n-1}{2} \\ 2t, & \text{if } i = \frac{n+1}{2} \\ t, & \text{if } i = \frac{n+3}{2} \\ 1, & \text{for all other } i \end{cases}$$

The ratio of Ball curves up to degree 9 is shown in Table 2:

Table 2: Ratio property of Ball curve for different degrees

Ratio	$\frac{(\beta_{1,n})^2}{\beta_{0,n}\beta_{2,n}}$	$\frac{(\beta_{2,n})^2}{\beta_{1,n}\beta_{3,n}}$	$\frac{(\beta_{3,n})^2}{\beta_{2,n}\beta_{4,n}}$	$\frac{(\beta_{4,n})^2}{\beta_{3,n}\beta_{5,n}}$	$\frac{(\beta_{5,n})^2}{\beta_{4,n}\beta_{6,n}}$	$\frac{(\beta_{6,n})^2}{\beta_{5,n}\beta_{7,n}}$	$\frac{(\beta_{7,n})^2}{\beta_{6,n}\beta_{8,n}}$	$\frac{(\beta_{8,n})^2}{\beta_{7,n}\beta_{9,n}}$
Terms	R_1^n	R_2^n	R_3^n	R_4^n	R_5^n	R_6^n	R_7^n	R_8^n
$n=3$	$2(1-t)$	$2t$						
$n=4$	$(1-t)^2$	4	t^2					
$n=5$	$1-t$	$2(1-t)$	$2t$	t				
$n=6$	1	$(1-t)^2$	4	t^2	1			
$n=7$	1	$1-t$	$2(1-t)$	$2t$	t	1		
$n=8$	1	1	$(1-t)^2$	4	t^2	1	1	
$n=9$	1	1	$(1-t)$	$2(1-t)$	$2t$	t	1	1
$\vdots$								

In a similar fashion, the Ball curve's ratio properties can be deduced as follows.

1. Symmetry: For all degrees n, the sequence of ratios is symmetric. That is, $R_i^n = R_{n-i}^n$ for $i = 1, 2, \ldots, n-1$.
2. Central Term: For even degrees ($n = 4, 6, 8, ...$), the central term is always 4.
3. Complementary Terms: For even degrees ($n = 4, 6, 8, ...$), the central term is always 4. The terms equidistant from the center complement each other to form a polynomial in t:
   - For odd degrees: $(1-t)$ and t
   - For even degrees: $(1-t)^2$ and t^2
4. Degree-Dependent Patterns:
   - For $n = 3$: The terms are $2(1-t)$ and $2t$.
   - For $n = 4$: The terms are $(1-t)^2$, 4, t^2.
   - For $n \geq 5$: The sequence starts and ends with 1, and the middle terms follow a pattern based on whether n is odd or even.
5. Increasing Stability: As the degree increases, more terms at the beginning and end of the sequence become stable at 1.

Shape uniqueness theorem for Rational Bezier curves:

Rational Bezier curves not only extends the design capabilities of traditional quadratic Bezier curves but also allows for complex shape manipulations through the incorporation of weights. Specifically, degree two rational Bezier curve can represent various conic sections parabolas, hyperbolas, and ellipses based on the relationship between the weights as shown in Eq. (2.4):

$$\text{Conic} = \begin{cases} \text{Parabola}, & \text{if } w_1^2 - w_0.w_2 = 0 \\ \text{Hyperbola}, & \text{if } w_1^2 - w_0.w_2 > 0 \\ \text{Ellipse}, & \text{if } w_1^2 - w_0.w_2 < 0. \end{cases} \tag{2.4}$$

For rational Bezier curves, the formula needs to account for the weights associated with each control point. The general formula for the ratio property of rational Bezier curves can be expressed as follows:

Definition 3: For $1 \leq i \leq n-1$, the general formula for the ratio property of Rational Bezier curve can be expressed as follows:

$$R_i^n = \frac{(R_{i,n})^2}{(R_{i-1,n}) \cdot (R_{i+1,n})} = \frac{(i+1)(n-i+1)}{i(n-i)} \cdot \frac{w_i^2}{w_{i-1}w_{i+1}} \tag{2.5}$$

By deduction, the similar properties of rational Bezier's ratio can be obtained, which is clearly shown in the full paper.

Application of shape uniqueness theorem:

In this section, we will showcase the conversion of the C-Bezier curve to a rational quadratic Bezier curve. Initially Zhang [4, 5] investigated cubic C-Bezier curves with one parameter α in the space span$\{1, t, \cos t, \sin t\}$. Later, Chen and Wang [12] constructed a C-Bezier curve of degree n for the space span$\{1, t, t^2, t^3, \ldots, t^{n-2}, \cos t, \sin t\}$, using an integral approach. These bases share the same properties as the Bernstein basis when the parameter $\alpha \to 0$. The blending functions for quadratic C-Bezier curve are defined as:

$$\begin{aligned} u_{02}(t) &= \frac{1 - cos(\alpha - t)}{1 - cos\alpha}, \\ u_{12}(t) &= \frac{1 - cost - cos(\alpha - t) + cos\alpha}{1 - cos\alpha}, \\ u_{22}(t) &= \frac{1 - cost}{1 - cos\alpha}. \end{aligned} \tag{2.6}$$

where $\alpha \in (0, t)$, $t \in [0, \alpha]$ and $0 < \alpha \leq \pi$. From the SUT, the shape of C-Bezier can be directly converted into rational quadratic Bezier curve for the same control points by equating their ratios, where $\{f_0, f_1, f_2\}$ are the blending functions of quadratic rational Bezier curves (RHS) and $\{u_{02}, u_{12}, u_{22}\}$ are the blending functions of C-Bezier curve defined in Eq. (2.6) (RHS):

$$\begin{aligned} \frac{f_1(t)^2}{f_0(t).f_2(t)} &= \frac{u_{12}(t)^2}{u_{02}(t).u_{22}(t)} \\ \therefore w_1 = \pm \frac{\sqrt{w_0}\sqrt{w_2}\sqrt{\cos\alpha + 1}}{\sqrt{2}}&, \; w_0 w_2 \neq 0 \end{aligned} \tag{2.7}$$

The conic type depends on the selection of weight values where the term that classifies the conic section is : $\Delta = w_1^2 - w_0 w_2$ as shown in Eq. (2.4) Substituting for w_1 in Eq. (2.6):

$$\Delta = \left(\frac{\sqrt{w_0}\sqrt{w_2}\sqrt{\cos\alpha + 1}}{\sqrt{2}} \right)^2 - w_0 w_2 = w_0 w_2 \left(\frac{\cos\alpha - 1}{2} \right) \tag{2.8}$$

The weights are always positive whereas $(cos\ \alpha - 1) < 0$ for $0 < \alpha \leq \pi$. Since Δ is always negative, the C-Bezier curve as an ellipse. We can solve Eq.(2.7) in terms of C-Bezier's shape parameter α as $\alpha = \cos^{-1}\left(\frac{2{w_1}^2}{w_0 w_2} - 1\right)$. This equation can be used to directly generate C-Bezier curves with chosen weight parameters.

Conclusions:

Shape Uniqueness Theorem has yielded a general pattern in the context of ratio properties for Bezier, Rational Bezier and Ball curves. Using SUT, we successfully transformed quadratic C-Bezier curve into rational quadratic Bezier form and it also aided in conic type identification. This research opens avenues for future studies in more non-polynomial curve types. By enhancing the theoretical understanding of the ratios of free-form curve's blending function , this approach could derive advancements in curve and surface modeling. Future work include devising a method to convert those curves with ratios in the form of parameter t. It is also worth investing the ratios of Timmer curve[13] which does not satisfy convex hull property.

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