Title:
Operations on Signed Distance Function Estimates

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Introduction:

Our paper presents a general theoretical framework to investigate the quantitative aspects of bounding distance functions. We propose a precision definition that quantifies the accuracy of the min/max representation of set-theoretic operations [5] in the entire space and demonstrate how the precision and the geometric configuration of the arguments determine the accuracy of the resulting approximation. Our theorems can be applied in an arbitrary geometrical context, e.g., for objects with or without volumes, implicit curves, non-differentiable or non-manifold surfaces, fractals, and any combination of these.

We identify a subset of Hart's signed distance lower bounds [3] called signed distance function estimates (SDFE) and show that the sphere tracing algorithm retains convergence under set-theoretic union and intersection operations, a result for which a general derivation has not yet been presented. Most so-called *distance estimates* used by the industry and the online creative coding communities such as ShaderToy are SDFEs, placing no practical restrictions on the applicability of our results. This paper builds upon the theoretical results of Luo et al. [4], Bálint et al.[1], and Valasek et al. [6].

Preliminaries:

Let us denote the $r > 0$ neighborhood of an $\boldsymbol{p} \in \mathbb{R}^3$ point with the $\mathcal{K}_r(\boldsymbol{p}) := \left\{\boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; d(\boldsymbol{x},\boldsymbol{p}) < r\right\}$ open set. For any $D \subseteq \mathbb{R}^3$ the radius $r \geq 0$ closed *offset set* from D is defined as $\overline{\mathcal{K}}_r(D) := \{\boldsymbol{x} \in \mathbb{R}^3 \,|\, d(\boldsymbol{x}, D) \leq r\}$. Similarly, the interior of $\overline{\mathcal{K}}_r(D)$ is denoted as $\mathcal{K}_r(D) := \operatorname{int} \overline{\mathcal{K}}_r(D)$. This equals $\{\boldsymbol{x} \in \mathbb{R}^3 \,|\, d(\boldsymbol{x}, D) < r\}$ if $r > 0$. The difference compared to the neighboorhood is to allow $r = 0$ which opens the set. Offsetting is additive $\mathcal{K}_{r_1}\big(\mathcal{K}_{r_2}(D)\big) = \mathcal{K}_{r_1+r_2}(D)$ $(r_1, r_2 > 0)$ due to Theorem 1 from [1]. Note that our definition differs from the offset surface defined with translations along the normal because we may not have a normal or a surface. Therefore, the offset is the neighbourhood, and the offset surface is the boundary of that neighbourhood.

Sphere Tracing:

Let us consider surfaces defined by an $f : \mathbb{R}^3 \to \mathbb{R}$ implicit function, such that the surface is the $\{f = 0\} := \{\boldsymbol{x} \in \mathbb{R}^3 \mid f(\boldsymbol{x}) = 0\}$ level-set. In this representation, given a ray with $\boldsymbol{s}(t) = \boldsymbol{p} + t \cdot \boldsymbol{v} \in \mathbb{R}^3$ with $\boldsymbol{v} \in \mathbb{R}^3$ unit-length direction and $\boldsymbol{p} \in$ origin point, we can define the ray-surface intersection problem

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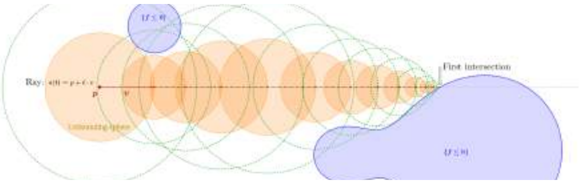

(a) Sphere tracing takes at most distance sized steps so that it does not overstep a solution and converges quickly. The unbounding spheres (orange circles) contain no surface points while each of the green spheres (circles) do, so f is an SDFE.

In : $\boldsymbol{p}, \boldsymbol{v} \in \mathbb{R}^3, \|\boldsymbol{v}\|_2 = 1$ ray, $f : \mathbb{R}^3 \to \mathbb{R}$ distance function
Out: $t \in [0, +\infty)$ distance traveled along the ray
$t := 0; \quad i := 0;$
for $i < i_{max}$ ***and*** $f(\boldsymbol{p} + t \cdot \boldsymbol{v})$ *not too small;* $i := i + 1$ **do**
| $t := t + f(\boldsymbol{p} + t \cdot \boldsymbol{v})$
end

(b) Basic sphere tracing adapted from [3].

Fig. 1: Sphere tracing visualized in 2D (a) is a practical algorithm (b) for implicit surface rendering.

as finding the smallest root of $f(\boldsymbol{s}(t))$ for $t > 0$. Let us define the *distance operator* $\mathbf{D} : \mathcal{P}(\mathbb{R}^3) \setminus \{\emptyset\} \to C(\mathbb{R}^3, [0, +\infty))$ as $\mathbf{D}_A(\boldsymbol{p}) := d(\boldsymbol{p}, A) := \inf_{\boldsymbol{a} \in A} \|\boldsymbol{p} - \boldsymbol{a}\|_2 \quad (\emptyset \neq A \subseteq \mathbb{R}^3, \boldsymbol{p} \in \mathbb{R}^3)$, where $C(\mathbb{R}^3, [0, +\infty))$ denotes the set of continuous functions from $\mathbb{R}^3$ to $[0, +\infty)$, and $\mathcal{P}(\mathbb{R}^3)$ is the power set of $\mathbb{R}^3$. This operator denotes the implicit distance function representation for any set of points in space, including curves and surfaces. Luo et al. investigate the singed distance operator in more detail in their paper [4].

Definition 1. $f : \mathbb{R}^3 \to \mathbb{R}$ is a *distance function* (DF) if $f = \mathbf{D}_{\{f=0\}}$.

Considering that for any $\boldsymbol{p} \in \mathbb{R}^3$ point, the surface is at least $f(\boldsymbol{p})$ distance away, meaning that we can take this distance-sized step along the ray without overstepping a solution. Sphere tracing in Algorithm 1b iteratively takes these steps along the ray.

Signed Distance Functions:

Definition 2 (SDF)**.** If $f : \mathbb{R}^3 \to \mathbb{R}$ function is continuous and $|f|$ is a distance function, then f is a **s**igned **d**istance **f**unction.

Here, the distance values of the function are augmented with a sign. From the perspective of the representation, this means that $\{f < 0\} := \{\boldsymbol{x} \in \mathbb{R}^3 \mid f(x) < 0\}$ is the "inside" and the $\{f > 0\} = \{-f < 0\}$ is the "outside" of the geometry. In this paper, we mean inside and outside as such, and the argument object will mean the set $\{f \leq 0\} = \{f < 0\} \cup \{f = 0\}$. For example, $\mathbb{R}^3 \ni \boldsymbol{p} \mapsto \|\boldsymbol{p}\|_2 - 1 \in [-1, +\infty)$ is a signed distance function of the closed unit sphere. Note that an SDF may not define an inside region, only the surface, because distance functions are SDFs as well.

Continuity in the definition is required to ensure that the SDFs are Bolzano functions, i.e. the signs do not change without crossing the surface. However, this does not imply that the signs have to change at $\{f = 0\}$, so distance functions are SDFs without interior ($\{f < 0\} = \emptyset$). Moreover, the definition implies that $\{f = 0\} \neq \emptyset$. Mathematically, the exact distance representations are important, and there are extensive studies that investigate signed distance functions [1], boundary projections [6], or both [4]. Practically, however, exact SDFs are infeasible for anything but the most trivial scenes.

A common way of obtaining a distance estimate from an implicit function is to divide it by one of its Lipschitz constants. We can generalize this to functions that are not Lipschitz continuous by using a more general quantity than the Lipschitz constants. To identify the necessary properties of this quantity, we derive an alternative definition to Hart's signed distance lower bounds from [3]. First, let us define the set of Lipschitz constants for any $f : \mathbb{R}^3 \to \mathbb{R}$ function as

$$\operatorname{Lip} f := \left\{ L > 0 \;\middle|\; \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3 : |f(\boldsymbol{x}) - f(\boldsymbol{y})| \leq L \cdot \|\boldsymbol{x} - \boldsymbol{y}\|_2 \right\}. \tag{2.1}$$

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Note that smallest Lipschitz constant of an SDF is 1, that is $\inf \operatorname{Lip} f = \min \operatorname{Lip} f = 1$.

Definition 3 (Closer factor)**.** For any $f : \mathbb{R}^3 \to \mathbb{R}$ function, we define the set of closer factors as

$$\mathcal{C}\, f := \left\{ Q > 0 \;\middle|\; |f| \leq Q \cdot \mathbf{D}_{\{f=0\}} \right\} \subseteq (0, \infty)$$

We interpret the above less symbol element-wise. This means that the $\mathcal{C}\, f$ is the set of positive numbers that scale the true distance function such that it is still larger than $|f|$ at every point. $\mathcal{C}\, f$ can be derived from the Lipschitz constant definition by restricting $\boldsymbol{y}$ in Equation 2.1 such that $\boldsymbol{y} \in \{f = 0\}$. Note that the Lipschitz continuity ($\operatorname{Lip} f \neq \emptyset$) is a much stronger requirement then having a non-empty closer factor set, i.e. $\mathcal{C}\, f \neq \emptyset$.

Then, we can define *signed distance lower bounds* consistently with Hart [3] if $f : \mathbb{R}^3 \to \mathbb{R}$ is such that $1 \in \mathcal{C}\, f$ and $\operatorname{sgn} \circ f \in C(\{f \neq 0\})$. The original definition only had the $1 \in \mathcal{C}\, f$ condition, which ensures that f is a lower bound to the actual distance. We also stipulate that $\operatorname{sgn} \circ f \in C(\{f \neq 0\})$, so that the resulting function only changes sign on the surface. Hence, the this condition guarantees inside $\{\operatorname{sgn} \circ f = -1\}$ and outside $\{\operatorname{sgn} \circ f = 1\}$ makes sense in relation to the surface without restricting geometric properties. As Hart noted in [3], we can generate a signed distance lower bound by dividing any Lipschitz continuous function with any of its Lipschitz constants. The same is true for closer factors, but the function need not be Lipschitz continuous.

Signed Distance Function Estimate:

This section introduces SDFEs, a set of signed distance bounds that possess convergence guarantees for algorithms such as sphere tracing by bounding their worst case slowdown. To quantify this, we define

Definition 4 (Farther factors)**.** For any $f : \mathbb{R}^3 \to \mathbb{R}$, let the set of farther factors be

$$\mathcal{F}\, f := \left\{ q > 0 \;\middle|\; |f| \geq q \cdot \mathbf{D}_{\{f=0\}} \right\} \subseteq (0, +\infty)\ .$$

Note that compared to closer factors the relation sign is flipped meaning f is increasing at least q times more further away from the surface than the distance does. The $\mathcal{F}\, f$ set is **unrelated to Lipschitz continuity** as it bounds the argument function from below with the actual distance.

Definition 5 (SDFE)**.** The function $f : \mathbb{R}^3 \to \mathbb{R}$ is a *signed distance function estimate* if $\mathcal{F}\, f \neq \emptyset$, $1 \in \mathcal{C}\, f$, and $\operatorname{sgn} \circ f \in C(\{f \neq 0\})$

Note that SDFs are SDFEs as well, since $\{1\} = \mathcal{F}\, f \cap \mathcal{C}\, f = (0, 1] \cap [1, \infty)$. We call any $q \in \mathcal{F}\, f$ **a precision of** f since $0 < q \leq 1$ quantifies the difference between an exact SDF and our estimate as demonstrated by Figure 2. Precision is also the maximum slowdown of the sphere tracing algorithm.

Set-operations:

For all theorems that follow, let f and g denote signed distance function estimates (SDFEs) with precisions $q_f \in \mathcal{F}\, f$ and $q_g \in \mathcal{F}\, g$, respectively. Let us also use the notational shorthands $f_0^- := \{f \leq 0\}$ and $f^- := \{f < 0\}$. The f_0^+ and f^+ symbols are analogous. Minimum and maximum on functions are to be interpreted element-wise.

The most important theorem in the field comes from [3] that states how set-operations can be applied to objects defined by SDFs. Adapting our notation, his theorem states that if $f, g : \mathbb{R}^3 \to \mathbb{R}$ are such that $1 \in \mathcal{C}\, f$ and $1 \in \mathcal{C}\, g$, then $1 \in \mathcal{C} \min(f, g)$ and $1 \in \mathcal{C} \max(f, g)$, and therefore:

Union: $h = \min(f, g)$ is a signed distance lower bound of the $f_0^- \cup g_0^-$ object.

Intersection: $h = \max(f, g)$ is a signed distance lower bound of $f_0^- \cap g_0^-$.

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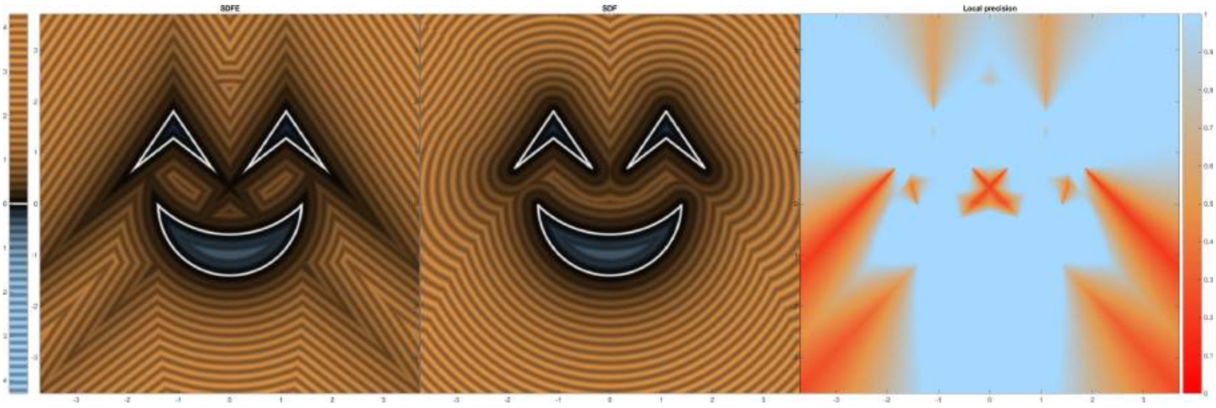

Fig. 2: Left: SDFE obtained through min and max set operations using transformations of a half-plane (line) and a circle primitive. The ratio of the SDFE (left) and the exact SDF (middle) is displayed on the right. The right image shows that the precision of the final SDFE is around 0.2 at maximum. We prove that there is a lower bound to the precision and thus a maximum slowdown for sphere tracing.

Difference: $h = \max(f, -g)$ is a signed distance lower bound of $f_0^- \setminus g_0^-$.

Despite the practical robustness, Hart's set theorems do not guarantee sphere tracing convergence, for example, the lower distance bound set may be empty. Figure 2 demonstrates how precision drops as the result of the above set operations. Since Hart has proved that h is a signed distance lower bound, we have to show that $\mathcal{F}\, h \neq \emptyset$.

Set-Contact Smoothness:

Our goal is to estimate the precision of the resulting $h = \max(f, g)$ function close to the exterior of the surface $\{h = 0\}$ without any geometric assumptions. However, the geometry of the intersection plays a vital role in the resulting precision. For this reason, We define the set-contact smoothness modulus as a function for $F, G \subseteq \mathbb{R}^3$ sets as

$$\sigma_{F,G}(\delta) := \min\left(\delta, d\left(F \setminus \mathcal{K}_{\frac{\delta}{2}}(F \cap G), G \setminus \mathcal{K}_{\frac{\delta}{2}}(F \cap G)\right)\right) \qquad (\delta \geq 0)\ .$$

For example, if F and G are two perpendicular intersecting lines, $\sigma_{F,G}(\delta) = \frac{\sqrt{2}}{2}\delta$. In general, $\sigma_{F,G}$ quantifies how smoothly F and G melds on various scales.

Proposition 1 (Properties)**.** *Let $F, G \subseteq \mathbb{R}^3$, then (i) $\sigma_{F,G}(0) = 0$, (ii) $\sigma_{F,G}$ is a monotonically increasing function, (iii) $\sigma_{F,G}(\delta) \leq \delta$, (iv) If F and G are closed sets, then $\forall\, \delta > 0\ :\ \sigma_{F,G}(\delta) \neq 0$.*

When one of the sets are not connected, the function $\sigma^*_{F,G}(\delta) := d\big(F \setminus_{\frac{\delta}{2}} G,\ G \setminus_{\frac{\delta}{2}} F\big) \quad (\delta \geq 0)$ can have a discontinuity and retain a higher value until $\mathcal{K}_\delta(F \cap G)$ reaches the next component. Therefore, the $\min(\delta, \cdot)$ is used in the equation allows the definition to make sense when $\sigma^*_{F,G}$ is infinite, and it ensures that properties (i) through (iv) hold.

Results:

The following theorem gives an almost global bound on the precision of the resulting SDFE.

Theorem 1 (Set operations)**.** *Suppose that f and $g : \mathbb{R}^3 \to \mathbb{R}$ are SDFEs, and let $0 < \delta \leq \operatorname{diam}\{f = 0\}$. Then, the following set-operations produce an SDFE with the function*

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Union $h = \min(f, g)$ *for the* $f_0^- \cup g_0^-$ *set if* f_0^+ *is bounded*

Intersection $h = \max(f, g)$ *for the* $f_0^- \cap g_0^-$ *set if* f_0^- *is bounded*

Difference $h = \max(f, -g)$ *for the* $f_0^- \setminus g_0^-$ *set if* f_0^- *is bounded*

with the precision

$$\frac{1}{4}\frac{\sigma_{\{f=0\},\{g=0\}}(\delta)}{\operatorname{diam}\{f=0\}} \cdot \min(q_f, q_g) \in \mathcal{F}\, h\Big|_{\mathbb{R}^3 \setminus \mathcal{K}_\delta(\{h=0\})} \tag{2.2}$$

Where $\operatorname{diam}\{f = 0\}$ is the diameter of one of the argument sets meaning that we must assume that it is bounded. The above theorem directly implies sphere tracing convergence on the resulting representation for the union, intersection, and difference operations. In practice, the convergence speed of the sphere tracing algorithm depends on the δ 'near-threshold' distance, on the diameter of the smaller object $\operatorname{diam} f_0^-$, and its SDFE bound K_f. The δ appears in sphere tracing implementations as an arbitrarily small value used for a distance threshold under which the computation is terminated. This way, sphere tracing stops when the surface is sufficiently approximated, i.e. the error is smaller than a pixel.

Conclusion:

We introduced a subset of signed distance lower bound functions called signed distance function estimates (SDFE) that have provable precision characteristics. These functions only pose constraints on the mapping and not on the represented geometry. We showed that sphere tracing retains convergence on SDFEs under set-theoretic operations. In particular, most so-called *distance estimates* used by the industry and online creative coding communities such as ShaderToy are SDFEs; thus the convergence guarantees derived here also hold for those constructs. These findings suggest that sphere tracing CSG trees composed of SDFEs may be optimized by reordering set-theoretic operations. This is subject to further research in conjunction with extending our analysis to various blending operations.

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