Title:
Rational Generalized Trigonometric Curve: Rationalization of Generalized Trigonometric Curve

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Keywords:
Generalized Trigonometric Curve, Circular Arcs, Elliptic Arcs, Uniqueness Theorem

DOI: 10.14733/cadconfP.2022.113-117

Introduction:
On the extensions of the cubic Bézier curve with four control points, to connect multiple segments with required continuity has been strongly intended and for example, tangent and curvature continuity at the start and end points are guaranteed independently by adding extra shape parameters. Contrary to this research trend, κ-curves, which control one curvature extremum on each curve segment instead of the end points, are defined as a sequence of the quadratic Bézier curve with three control points. The authors has proposed generalized trigonometric basis functions consisting of $(\sin t, \cos t, 1)$ and defined the generalized trigonometric curve in order to extend κ-curves [4]. In this study, we will show that the linear generalized trigonometric curve defined by three control points generates an elliptical arc, but cannot generate an arbitrary elliptical curve. Hence we will rationalize it to express an arbitrary elliptical arc as well as arbitrary arcs of parabola and hyperbola.

κ-Curve:
The κ-curve, proposed recently by [7], is an interpolating spline which is curvature-continuous almost everywhere and passes through input points at the local curvature extrema. It has been implemented as the curvature tool in Adobe Illustrator® and Photoshop® and is accepted as a favored curve design tool by many designers (see e.g. [1, 2]).

We consider the reasons for the success of κ-curve to be:

1. Information along contours is concentrated at local maxima of curvature.
2. Curves of low degree have smooth distribution of curvature.
3. G^2-continuous curves tend to look fairer than only G^1-continuous ones.

Generalized Trigonometric Curve:

In this section, we describe *the generalized trigonometric curve*. The blending functios of the curve are $\{u, v, w\}$ given by

$$\begin{aligned} u &= 1 - S, \\ v &= S(1-S) + C(1-C) = S + C - 1, \\ w &= 1 - C. \end{aligned} \tag{1}$$

where $S = \sin\frac{\pi t}{2}$, $C = \cos\frac{\pi t}{2}$, for $\alpha \in (0, 2)$, $t \in [0, 1]$. It is straightforward to define a curve by these blending functions with three control points, which we can regard as a "linear" generalized trigonometric curve since the highest degree the trigonometric functions are in is one.

One interesting relationship among these functions is

$$v^2 = 2uw, \tag{2}$$

which enables

$$(u + v + w)^2 = u^2 + 2uv + 4uw + 2vw + w^2, \tag{3}$$

and yields the five blending functions $\{u^2, 2uv, 4uw, 2vw, w^2\}$, associated with five control points. We can define a curve using these blending functions and regard it as a "quadratic" trigonometric curve since the highest power of each blending function is now degree two.

In a similar way, we can extend blending functions of "degree" n with $2n + 1$ control points. We can perform a recursive procedure called Gobithaasan-Miura's algorithm to evaluate a curve of any degree similar to de Casteljau's algorithm avoiding the overhead of trigonometric function evaluation. This means that it is not necessary to calculate the coefficients of blending functions, or keep a coefficient table.

In order to analyze what kind of curve can be generated by a linear generalized trigonometric curve, without loss of generality up to similarity, we specify its three control points as $(-1, 0)$, (b, h), and $(1, 0)$. When $h = 0$, the curve becomes a line segument on the x-axis and we assume that $h \neq 0$. Then the linear generalized trigonometric curve is given by

$$x = (b + 1)S + (b - 1)C - b \tag{4}$$

$$y = h(S + C - 1) \tag{5}$$

By using the above equations and $S^2 + C^2 = 1$ and eliminating S and C, the following equation is obtained:

$$h^2x^2 + (b^2 + 1)y^2 - 2b\,h\,xy + 2hy - h^2 = 0 \tag{6}$$

In the above euqation, the coefficients of x^2 and y^2 are $h^2 > 0$ and $b^2 + 1 > 0$, repsectively and this equation represents an ellipse [6]. Hence the linear generalized trigonometric curve represents an elliptical arc cut out from the ellipse. Because of symmetry of the circle, if the lengths of the two line segments connecting the control points are different, no circular arc is represented. Furthermore if we assume that the locations of the control points are made to be symmetrical along the y-axis by $b = 0$,

$$h^2x^2 + y^2 + 2hy - h^2 = \frac{1}{h^2}\left(x^2 + \frac{1}{h^2}(y^2 + h)^2 - 2\right) = 0 \tag{7}$$

This equation does not represent a circle except for $h = \pm 1$ as explained below. When $h = \pm 1$, the two line segments connecting the control points become the same length and orthogonal each other and the

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linear generalized trigonometric curve represents a quater circular arc. Therefore in order to express an arbinrary circular or ellipical arc, its rationalization is necessary.

Although the left side of equation (6) includes the term of y and a constant, we can elliminate them by translating the curve along the y-axis. Hence it is enough to analyze the following quadratic form:

$$h^2x^2 - 2b\,h\,xy + (b^2+1)y^2 = (\ x, y\ )M \begin{pmatrix} x \\ y \end{pmatrix} \tag{8}$$

where

$$M = \begin{pmatrix} h^2 & -bh \\ -bh & (b^2+1) \end{pmatrix} \tag{9}$$

The eigenvalues λ_0, $\lambda_1$of matric M are given by

$$\lambda_0 = \frac{1}{2}\left(b^2 + h^2 + 1 - \sqrt{b^2(b^2+2(h^2+1)) + (h^2-1)^2}\right) \tag{10}$$

$$\lambda_1 = \frac{1}{2}\left(b^2 + h^2 + 1 + \sqrt{b^2(b^2+2(h^2+1)) + (h^2-1)^2}\right) \tag{11}$$

Hence by applying an appropriate transformation, we obtain

$$\lambda_0 x^2 + \lambda_1 y^2 = r^2. \tag{12}$$

If $h \neq 0$, $\lambda_0 > 0$ and $\lambda_1 > 0$. Then the above equation represents an ellipse. Especially when $\lambda_0 = \lambda_1$, or since $b^2 + h^2 + 1 - \sqrt{b^2(b^2+2(h^2+1)) + (h^2-1)^2} = 0$, $b = 0$ and $h = \pm 1$. So this represents a circle. In this case, the linear generalized trigonometric curve becomes a quater circular arc. Even though we assume $b = 0$ and locate the control points symmetrically, some specific circular arc is represented and no arbitray circular arc is obtained. In the ellipse case, the number of the parameters of the implicit function expressing an ellise is essentially 5 and one degree of freedom remains by specifying the postions of the end points and tahgent vectors there (4 constraints). However the ratio of λ_0 and λ_1 is constrained as the circle, we cannot represent an arbitray circular and elliptical arc and we need its rationalization.

Figure 1 shows examples of the linear generalized trigonometric curve. To clarify its properties, we draw quadratic Bézier curves defined by the same control points at the same time. In Fig.1(a), the locations of the control points are $(0,0)$, $(1,1)$ and $(1,0)$. In (b) and (c), only the first control points are translated to $(0,1)$ and $(0,2)$. The generalied trigonomtric curves are drawn in blue and the quadratic Bézier curves in orange. From these figures, the generalized trigonometric curve has smaller absolute curvature and are more rounded than the Bézier curve. Especially in (b), the two line segments connecting the control points become the same length and orthogonal each other and its equation can be simplified as $(\sin\frac{\pi}{2}t, \cos\frac{\pi}{2}t)$. It's a quater circular arc.

Rational Quadratic Bézier Curve:

It is very common to represent a circular arc by a quadratic rational Bézier curve as

$$\boldsymbol{C}(t) = \frac{(1-t)^2\boldsymbol{P}_0 + 2(1-t)t\sigma\boldsymbol{P}_1 + t^2\boldsymbol{P}_2}{(1-t)^2 + 2(1-t)t\sigma + t^2} \tag{13}$$

where σ is a weight of $\boldsymbol{P}_1$. For example when $\boldsymbol{P}_0 = (-1, a)$, $\boldsymbol{P}_1 = (0, 0)$ and $\boldsymbol{P}_2 = (1, a)$ for a given a, if $\sigma = 1/\sqrt{a^2+1}$ the curve becomes a circular arc.

Hence we define a blending function $w(t)$ as follows:

$$w(t) = \frac{t^2}{(1-t)^2 + 2(1-t)t\sigma + t^2} \tag{14}$$

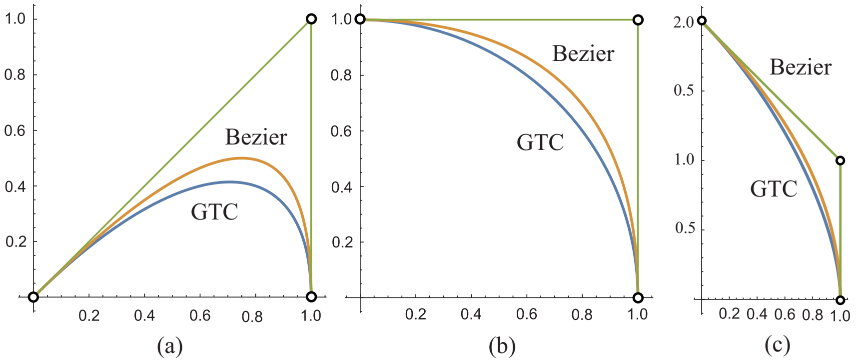

Fig. 1: Examples of linear GT curves with quadratic Bézier curves.

For this basis, the following equations is satisfied:

$$v(t)^2 = 4\sigma^2 u(t)w(t) \tag{15}$$

Uniqueness Theorem of the Shape of the Curve:

The authors have proved a theorem called uniqueness theorem of the shape of the curve [5]. We describe the theorem without proof. Please refer to [5] for its proof. We assume that for $0 \le t \le 1$ a curve $\boldsymbol{C}(t)$ is defined by three control points $\boldsymbol{P}_0$, $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ as

$$\boldsymbol{C}(t) = u(t)\boldsymbol{P}_0 + v(t)\boldsymbol{P}_1 + w(t)\boldsymbol{P}_2 \tag{16}$$

where $0 \le w(t) \le 1$, $0 \le v(t) \le 1$. If there is such a constant α that

$$v(t)^2 = \alpha u(t)w(t) \tag{17}$$

for $0 \le t \le 1$, then the following theorem is satisfied:

Theorem 1. *Uniqueness Theorem: The shape of the curve $\boldsymbol{C}(t)$ is determined by α exclusively and it does not depend on the basis functions $\{u(t), v(t), w(t)\}$ which are used to define the curve.*

Rational Generalized Trigonometric Curve:

Similar to the rational quadratic Bézier curve, with weight ω we define the rational linear generalized trigonometric curve as follows:

$$\boldsymbol{C}(t) = \frac{u(t)\boldsymbol{P}_0 + v(t)\omega\boldsymbol{P}_1 + w(t)\boldsymbol{P}_2}{u(t) + \omega v(t) + w(t)} \tag{18}$$

$$= u_r(t)\boldsymbol{P}_0 + v_r(t)\boldsymbol{P}_1 + w_r(t)\boldsymbol{P}_2 \tag{19}$$

where

$$u_r(t) = \frac{1-S}{u(t) + v(t)\omega + w(t)}, \tag{20}$$

$$v_r(t) = \frac{S+C-1}{u(t) + v(t)\omega + w(t)}, \tag{21}$$

$$w_r(t) = \frac{1-C}{u(t) + v(t)\omega + w(t)}. \tag{22}$$

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Then

$$v_r(t)^2 = 2\omega^2 \, u_r(t) \, w_r(t) \tag{23}$$

Therefore by comparing equations (15) and (23), and applying Uniquness theorem, when $\omega = \sqrt{2}\sigma$, the shapes of the linear generalized trigonometric curve and the quadratic Bézier curve are identical althogh their parametrizations are different. Therefore the rational linear generalized trigonometric curve can represent an arbitrary elliptical arc as well as arbitrary arcs of parabola and hyperbola. Please refer to [3] about conics as rational quadratics. Furthermore by the same reason, if we rationalize generalized hyperboic curve and splines in tension, they can represent an arbitrary elliptical arc as well as arbitrary arcs of parabola and hyperbola.

Conclusions:

We has shown that the linear generalized trigonometric curve defined by three control points generates an elliptical arc, but cannot generate an arbitrary elliptic curve. Hence we have rationalized it to express an arbitrary elliptical arc as well as arbitrary arcs of parabola and hyperbola. By the same reason, we have shown that the rational generalized hyperboic curve and rational splines in tension can represent an arbitrary elliptical arc as well as arbitrary arcs of parabola and hyperbola. In the future research we will investigate other properties of these rationalized curves.

Acknowledgement:

This work was supported by JST CREST Grant Number JPMJCR1911. It was also supported JSPS Grant-in-Aid for Scientific Research (B) Grant Number 19H02048, JSPS Grant-in-Aid for Challenging Exploratory Research Grant Number 26630038. The authors acknowledge the support by 2016, 2018 and 2019 IMI Joint Use Program Short-term Joint Research "Differential Geometry and Discrete Differential Geometry for Industrial Design" (September 2016, September 2018 and September 2019). The second author acknowledges University Malaysia Terengganu for approving sabbatical leave which was utilized to work on emerging researches, including this work.

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