Title:
Uniqueness Theorem on the Shape of Free-form Curves Defined by Three Control Points

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Introduction:
On the researches of free-form curves, those of the quadratic curve have been become very active because of κ-curves[1, 2]. In this paper, we will prove the uniqueness theorem on the shape of free-form curves defined by three control points, including quadratic non-rational: integral and rational Bézier curves, generalized trigonometric and hyperbolic curves and splines in tension [3].

Generalized Trigonometric Basis:
In this section, we describe our new *generalized trigonometric basis*. This is based on the trigonometric cubic Bernstein-like basis [4], which we are going to review first.

The trigonometric cubic Bernstein-like basis functions have an extra shape parameter α, and are defined by

$$\begin{aligned} f_0 &= \alpha S^2 - \alpha S + C^2 = 1 + (\alpha - 1)S^2 - \alpha S, \\ f_1 &= \alpha S(1 - S), \\ f_2 &= \alpha(S^2 + C - 1) = \alpha C(1 - C), \\ f_3 &= (1 - \alpha)S^2 - \alpha C + \alpha = 1 + (\alpha - 1)C^2 - \alpha C, \end{aligned} \tag{2.1}$$

where $S = \sin\frac{\pi t}{2}$, $C = \cos\frac{\pi t}{2}$, for $\alpha \in (0, 2)$, $t \in [0, 1]$. Note that these functions satisfy partition of unity, i.e., $\sum_{i=0}^{3} f_i(t) = 1$ for any α. When $\alpha = 1$, the above functions are simplified to

$$\begin{aligned} f_0 &= 1 - S, \\ f_1 &= S(1 - S), \\ f_2 &= C(1 - C), \\ f_3 &= 1 - C. \end{aligned} \tag{2.2}$$

If we add the second and third functions together and rename them to u, v and w, we obtain blending functions $\{u, v, w\}$ as follows:

$$\begin{aligned} u &= 1 - S, \\ v &= S(1-S) + C(1-C) = S + C - 1, \\ w &= 1 - C. \end{aligned} \tag{2.3}$$

It is straightforward to define a curve by these blending functions with three control points, which we can regard as a "linear" trigonometric curve since the highest degree the trigonometric functions are in is one.

One interesting relationship among these functions is

$$v^2 = 2uw, \tag{2.4}$$

which enables

$$(u + v + w)^2 = u^2 + 2uv + 4uw + 2vw + w^2, \tag{2.5}$$

and yields the five blending functions $\{u^2,\ 2uv,\ 4uw,\ 2vw,\ w^2\}$, associated with five control points. We can define a curve using these blending functions and regard it as a "quadratic" trigonometric curve since the highest power of each blending function is now degree two.

In a similar way, we can extend blending functions of "degree" n with $2n + 1$ control points. As explained in Appendix, we can perform a recursive procedure to evaluate a curve of any degree similar to de Casteljau's algorithm avoiding the overhead of trigonometric function evaluation. We call this procedure Gobithaasan-Miura's recursive algorithm. This means that it is not necessary to calculate the coefficients of blending functions, or keep a coefficient table. The coefficients of the generalized trigonometric curve are listed as an triangle as Pascal's triangle and we call it Miura's triangle as shown in the Appendix.

Rational Quadratic Bernstein Basis:

It is very common to represent a circular arc by a quadratic rational Bézier curve as

$$\boldsymbol{C}(t) = \frac{(1-t)^2 \boldsymbol{P}_0 + 2(1-t)t\sigma \boldsymbol{P}_1 + t^2 \boldsymbol{P}_2}{(1-t)^2 + 2(1-t)t\sigma + t^2} \tag{2.6}$$

where σ is a weight of $\boldsymbol{P}_1$. For example when $\boldsymbol{P}_0 = (-1, a)$, $\boldsymbol{P}_1 = (0, 0)$ and $\boldsymbol{P}_2 = (1, a)$ for a given a, if $\sigma = 1/\sqrt{a^2 + 1}$ the curve becomes a circular arc.

Hence we define a blending function $w(t)$ as follows:

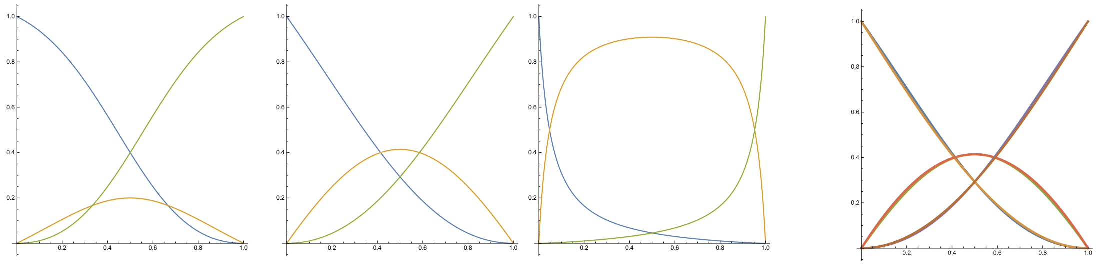
$$w(t) = \frac{t^2}{(1-t)^2 + 2(1-t)t\sigma + t^2} \tag{2.7}$$

For this basis, the following equations is satisfied:

$$v(t)^2 = 4\sigma^2 u(t) w(t) \tag{2.8}$$

Figure 1(a) shows graphs of $\{u(t), v(t), w(t)\} = \{w(1-t), 1 - w(1-t) - w(t), w(t)\}$ for $\sigma = 1/4$, $1/2$, $1/\sqrt{2}$, 2 and 10. By increasing σ, a curve defined by these basis functions approaches to a polyline connecting its control points.

Note that if $\sigma = 1$, since the basis becomes that of the non-rational quadratic Bernstein basis, $\alpha = 4$. If $\sigma = 1/\sqrt{2}$, $\alpha = 2$. However $w(t) \neq 1 - \cos(\pi t/2)$. Figure 1(b) compares these two basis functions and they are very similar, but not indentical.



σ=1/4 σ=1/sqrt(2) σ=10

(a) (b)

Fig. 1: (a) Rational quadratic Bernstein basis functions, (b) Comparison between the rational quadratic Bernstein basis functions and $\{1-\sin(\pi t/2), \sin(\pi t/2)+\cos(\pi t/2)-1, 1-\cos(\pi t/2)\}$

Since there are two types of the bases whose $\alpha = 2$, the conditions

$$\{1-w(t)-w(1-t)\}^2 = \alpha w(t)w(1-t) \tag{2.9}$$

for a given constant $\alpha > 0$ with $w(0) = 0$, $w(1) = 1$ and $dw(0)/dt = 0$ do not determine function $w(t)$ uniquely.

Notice that when $t = 1/2$, from the following equation:

$$\begin{aligned} (1-2w(\frac{1}{2}))^2 &= \alpha w(\frac{1}{2})^2 \\ (4-\alpha)w(\frac{1}{2})^2 - 4w(\frac{1}{2}) + 1 &= 0 \end{aligned} \tag{2.10}$$

When $\alpha = 4$, $w(1/2) = 1/4$. Since $0 < w(1/2) < 1$, when $\alpha < 4$, $w(1/2) = (2-\sqrt{\alpha})/(4-\alpha)$ and when $\alpha > 4$, $w(1/2) = (\sqrt{\alpha}-2)/(\alpha-4)$. Therefore although the basis functions are different, if they have the same α value, when $t = 1/2$, the values of these basis functions are exactly the same.

Uniqueness Theorem of the Shape of the Curve:

We will prove a theorem called uniqueness theorem of the shape of the curve. We assume that for $0 \le t \le 1$ a curve $\boldsymbol{C}(t)$ is defined by three control points $\boldsymbol{P}_0$, $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ as

$$\boldsymbol{C}(t) = u(t)\boldsymbol{P}_0 + v(t)\boldsymbol{P}_1 + w(t)\boldsymbol{P}_2 \tag{2.11}$$

where $0 \le w(t) \le 1$, $0 \le v(t) \le 1$ and

$$\begin{aligned} u(t)+v(t)+w(t) &= 1 \\ u(t) &= w(1-t) \\ w(0) &= 0 \\ w(1) &= 1 \\ \frac{dw(t)}{dt} &> 0 \quad \text{for} \quad 0 < t < 1 \end{aligned} \tag{2.12}$$

If there is such a constant α that

$$v(t)^2 = \alpha u(t)w(t) \tag{2.13}$$

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for $0 \le t \le 1$, then the following theorem is satisfied:

Theorem 1. *Uniqueness Theorem: The shape of the curve $\boldsymbol{C}(t)$ is determined by α exclusively and it does not depend on the basis functions $\{u(t), v(t), w(t)\}$ which are used to define the curve.*

Proof. For a given value $w_0 = w(t_0)$, $0 \le w_0 \le 1$, let $u_0 = u(t_0)$. Since $v(t) = 1 - u(t) - w(t)$,,

$$(1 - u_0 - w_0)^2 = \alpha u_0 w_0 \tag{2.14}$$

Hence

$$u_0 = \frac{(\alpha - 2)w_0 + 2 - \sqrt{\alpha w_0((\alpha - 4)w_0 + 4)}}{2} \tag{2.15}$$

Since u_0 is uniquely determined by w_0, the location of the point $\boldsymbol{C}(t_0)$ is also uniquely determined because $\{u(t), v(t), w(t)\}$ are barycentric coordinates of triangle $\boldsymbol{P}_0\boldsymbol{P}_1\boldsymbol{P}_2$. By changing t from 0 to 1, $w(t)$ also increases from 0 to 1 and the shape of the curve $\boldsymbol{C}(t)$ is also completely determined. □

Figure 2 shows u_0 for $0 < w_0 < 1$ and $0 < \alpha < 10$

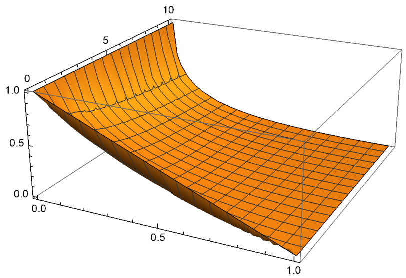

Fig. 2: u_0 for $0 < w_0 < 1$ and $0 < \alpha < 10$

Conclusions:

We have proved the uniqueness theorem on the shape of free-form curves defined by three control points, including quadratic non-rational and rational Bézier curves, generalized trigonometric and hyperbolic curves and splines in tension. We have also shown that we can generate infinite different Miura's triangles for different α values. For future work, we would like to extend our theorem for higher-degree free-from curves.

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References:

[1] Yan, Z.; Schiller, S.; Wilensky G.;, Carr, N.; Schaefer, S.: κ-curves: interpolation at local maximum curvature. ACM Transactions on Graphics 36, 4 (2017), Article 129. https://doi.org/10.1145/3072959.3073692

[2] Yan, Z.; Schiller, S.; Schaefer, S.: Circle reproduction with interpolatory curves at local maximal curvature points. Computer Aided Geometric Design 72, June 2019, Pages 98-110,. https://doi.org/10.1016/j.cagd.2019.06.002

[3] Per Erik Koch and Tom Lyche, Construction of Exponential Tension B-splines of Arbitrary Order, in Curves and Surfaces, Editors: Pierre-Jean Laurent, Alain Le Méhauté, Larry L. Schumaker, Academic press, 1991. https://doi.org/10.1016/B978-0-12-438660-0.50039-X

[4] Usman, M.M., Abbas, M., Miura, K.T.: Some engineering applications of new trigonometric cubic Bézier-like curves to free-form complex curve modeling. Journal of Advanced Mechanical Design, Systems, and Manufacturing 14(4), 1 (2020). https:10.1299/jamdsm.2020jamdsm0048

Appendix: Generalized Trigonometric Basis Functions:

Gobithaasan and Miura's Recursive Algorithm For our new trigonometric basis, we can derive a recursive algorithm similar to de Casteljau's algorithm. For simplicity we explain only the quadratic case, but it can be extended to a general degree n by induction. To shorten expressions, we use $u = 1 - S(t)$, $v = S(t) + C(t) - 1$ and $w = 1 - C(t)$, where $S(t) = \sin\frac{\pi t}{2}$ and $C(t) = \cos\frac{\pi t}{2}$. Note that $v^2 = 2uw$, and

$$\begin{aligned}(u+v+w)^2 &= \\ &u(u+v+w) + v(u+v+w) + w(u+v+w).\end{aligned} \tag{2.16}$$

For a quadratic curve with this basis, five control points P_i ($i = 0 \dots 4$) are used, and the curve point at t is evaluated as

$$\begin{bmatrix} u & v & w \end{bmatrix} \begin{bmatrix} P_0 & P_1 & P_2 \\ P_1 & P_2 & P_3 \\ P_2 & P_3 & P_4 \end{bmatrix} \begin{bmatrix} u & & \\ & v & \\ & & w \end{bmatrix}. \tag{2.17}$$

Hence the algorithm repeats a simple blending of three points $uP_{i-1} + vP_i + wP_{i+1}$ to generate a point on the given curve.

Miura's Triangle: We can also construct a triangle using the coefficients of trigonometric basis functions, similarly to Pascal's triangle. Below is a table of degree elevation, from the first row representing degree 1 to the sixth row representing degree 6:

$$\begin{array}{ccccccccccccc}
 & & & & & 1 & 1 & 1 & & & & & \\
 & & & & 1 & 2 & 4 & 2 & 1 & & & & \\
 & & & 1 & 3 & 9 & 8 & 9 & 3 & 1 & & & \\
 & & 1 & 4 & 16 & 20 & 34 & 20 & 16 & 4 & 1 & & \\
 & 1 & 5 & 25 & 40 & 90 & 74 & 90 & 40 & 25 & 5 & 1 & \\
1 & 6 & 36 & 70 & 195 & 204 & 328 & 204 & 195 & 70 & 36 & 6 & 1
\end{array} \tag{2.18}$$