



Title:

Vector Construction Techniques for Circle Packing

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Introduction:

Design problems such as routing wires or pipes through a wall, Figure 1(a), and routing wires through a slip ring core, Figure 1(b), both reduce to the problem of packing a given set of circles into a geometric shape while satisfying some minimization criterion. See [1,3] for examples and [2] for additional references. Depending on the geometry, the packing involves fitting a given circle tangent to two lines, tangent to a circle and a line, and tangent to two circles.

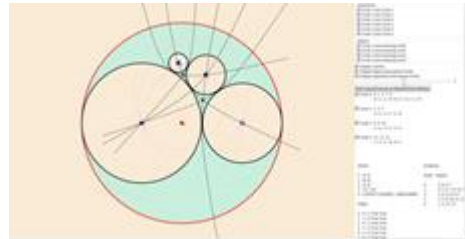
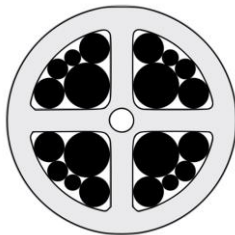
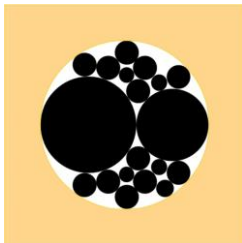


Fig. 1: (a) Wires through hole (b) Wires through slip ring core.

Fig. 2: Screen from test software.

The authors are developing test software to compare various packing algorithms and identify adjustments that would improve the packing, Figure 2. The software needs the center path arcs and the tangent vectors to the arcs as a function of the radius of the circle being fitted. The fittings are old problems that were solved for CAD systems by solving systems of equations [4,5]. These solutions could be used, but managing the cases is awkward and getting the tangent vector is even more awkward. The authors have developed compact and visually intuitive methods for the fittings and for generating the center arcs and tangent vectors. Although the derivations are tedious, the resulting formulas are simple and the method can be generalized to other CAD line and circle constructions and offers advantages over the traditional methods by being intuitive and simple to code and test.

Main Idea:

The main idea is to use vector-based constructions for a compact formulation and for some isolation from the underlying coordinate system, which is desirable since the problems are purely geometric.

Notation, Conventions and Preliminaries

Points and non-unit vectors will be represented by a bold uppercase letter, **P**. For interactive usage, a point, \mathbf{P}_{pick} , picked from the graphics display by the user selects cases and subcases. A unit vector is

represented as $\hat{v} = (v_x, v_y)$, and its orthogonal complement, as $\hat{v}^\perp = (-v_y, v_x)$. A unit vector from point \mathbf{Q} to point \mathbf{P} is $\hat{v} = (\mathbf{P} - \mathbf{Q}) / \|\mathbf{P} - \mathbf{Q}\|$, where $\|\mathbf{P} - \mathbf{Q}\|$ is the distance between \mathbf{Q} and \mathbf{P} .

Lines $Ax + By = C$ are in unit-normal distance form, i.e. $A^2 + B^2 = 1$ and C is the signed distance from the origin to the line. If the point $\mathbf{P} = (x, y)$ is on a line with unit normal $\hat{n} = (A, B)$, then the equation of the line is $\hat{n} \cdot \mathbf{P} = C$. The orthogonal complement of \hat{n} , \hat{n}^\perp , is parallel to the line.

In some constructions, multiple cases and subcases are organized via the following functions.

$$\text{signum}(x) = \begin{cases} +1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases} \quad \text{signum}_+(x) = \begin{cases} +1 & x \geq 0 \\ -1 & x < 0 \end{cases} \quad \text{signum}_-(x) = \begin{cases} +1 & x > 0 \\ -1 & x \leq 0 \end{cases}$$

If \mathbf{P} , \mathbf{P}_1 and \mathbf{P}_2 are collinear points with $\mathbf{P}_1 \neq \mathbf{P}_2$, then \mathbf{P} can be expressed as a linear combination of \mathbf{P}_1 and \mathbf{P}_2 . Compute the distances (norms) $\lambda_1 = \|\mathbf{P}_1 - \mathbf{P}\|$ and $\lambda_2 = \|\mathbf{P}_2 - \mathbf{P}\|$. If \mathbf{P} is between \mathbf{P}_1 and \mathbf{P}_2 , then $\mathbf{P} = (\lambda_2 \mathbf{P}_1 + \lambda_1 \mathbf{P}_2) / (\lambda_1 + \lambda_2)$. If \mathbf{P}_1 is between \mathbf{P} and \mathbf{P}_2 , then $\mathbf{P} = (\lambda_2 \mathbf{P}_1 - \lambda_1 \mathbf{P}_2) / (-\lambda_1 + \lambda_2)$. If \mathbf{P}_2 is between \mathbf{P}_1 and \mathbf{P} , then $\mathbf{P} = (-\lambda_2 \mathbf{P}_1 + \lambda_1 \mathbf{P}_2) / (\lambda_1 - \lambda_2)$.

One construction uses Heron's formula for the area of a triangle to compute the height h of a triangle. With sides as labelled in Figure 3, the semi-perimeter is $p_s = (a + b + c) / 2$. Then the area of the triangle is $A = \sqrt{p_s(p_s - a)(p_s - b)(p_s - c)} \Rightarrow A = \sqrt{(c^2 - (a + b)^2)((a - b)^2 - c^2)} / 4$, which gives the height of the triangle as $h = 2A / c = \sqrt{(c^2 - (a + b)^2)((a - b)^2 - c^2)} / (2c)$. Again referring to Figure 3, the Law of Cosines is $a^2 = b^2 + c^2 - 2bc \cos \theta \Rightarrow 2bc \cos \theta = b^2 + c^2 - a^2$, which implies $b^2 + c^2 - a^2 > 0 \Leftrightarrow 0 < \theta < \pi / 2$ and $b^2 + c^2 - a^2 < 0 \Leftrightarrow \pi / 2 < \theta < \pi$.

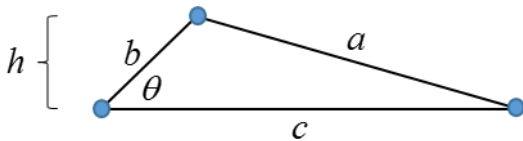


Fig. 3: General triangle.

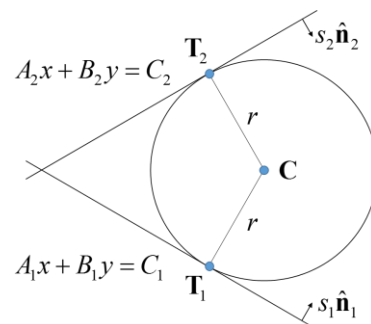


Fig. 4: Circle tangent to two lines.

Constructions:

Circle Tangent to Two Given Lines

This is the simplest of the circle fittings and shows the idea of the method but not its value. Let the given radius be r and let the normal vectors to the given lines be \hat{n}_1 and \hat{n}_2 . Define the values $s_1 = \text{signum}_+(\hat{n}_1 \cdot \mathbf{P}_{pick} - C_1)$ and $s_2 = \text{signum}_+(\hat{n}_2 \cdot \mathbf{P}_{pick} - C_2)$. (For the packing application, s_1 and s_2 are specified directly.) The intersecting lines divide the plane into four quadrants and $s_1 \hat{n}_1$ and $s_2 \hat{n}_2$ point into the quadrant containing \mathbf{P}_{pick} , and the constructed circle, as in Figure 4.

If line one is translated a distance r in the direction $s_1 \hat{n}_1$ and line two is translated a distance r in the direction $s_2 \hat{n}_2$, the center \mathbf{C} is at the intersection of the translated lines, i.e. $s_1 \hat{n}_1 \cdot \mathbf{C} = s_1 C_1 + r$ and

$s_2\hat{\mathbf{n}}_2 \cdot \mathbf{C} = s_2C_2 + r$. The equations are solved using Gaussian elimination with partial pivoting to give \mathbf{C} , which is used to compute the points of tangency as $\mathbf{T}_1 = \mathbf{C} - r s_1 \hat{\mathbf{n}}_1$ and $\mathbf{T}_2 = \mathbf{C} - r s_2 \hat{\mathbf{n}}_2$. The tangent vector to the center path arc is $\mathbf{T} = s_1 \hat{\mathbf{n}}_1 + s_2 \hat{\mathbf{n}}_2$.

Circle Tangent to a Given Line and Circle

Let the given radius be r . Let the given line be $Ax + By = C$ with unit normal $\hat{\mathbf{n}}$. Let the given circle have center \mathbf{C}_G and radius r_G . With $s = \text{signum}_+(\hat{\mathbf{n}} \cdot \mathbf{C}_G - C)$, the vector $s\hat{\mathbf{n}}$ points into the half space containing \mathbf{C}_G . The distance from the given line to the center of the given circle is $d = s(\hat{\mathbf{n}} \cdot \mathbf{C}_G - C)$. In the first case, Figure 5(a), $r_G \leq d$, the given circle does not intersect the given line (but could be tangent to it). In the second case, Figure 5(b), $r_G > d$, the given circle does intersect the given line.

Distinguish the upper and lower subcases using $s_{ul} = \text{signum}_+(\mathbf{P}_{pick} - \mathbf{C}_G) \cdot s\hat{\mathbf{n}}^\perp$. Additional subcases for Figure 5(b) will be identified using $s_1 = \text{signum}_+(\|\mathbf{P}_{pick} - \mathbf{C}_G\| - r_G)$, \mathbf{P}_{pick} inside or outside of the given circle, and $s_2 = \text{signum}_+(s(\hat{\mathbf{n}} \cdot \mathbf{P}_{pick} - C))$, \mathbf{P}_{pick} left or right of the given line.

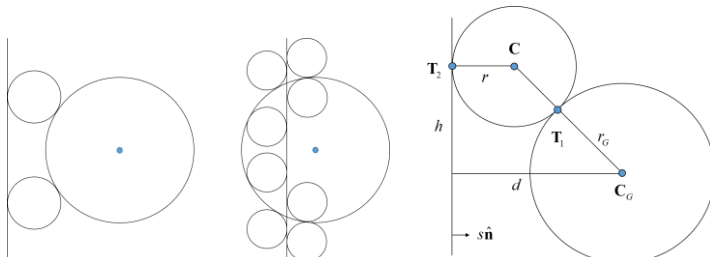


Fig. 5: Cases: (a) non intersecting (b) intersecting.

Fig. 6: $s_1 > 0, s_2 > 0$.

Case	Restriction	s_1	s_2
Figure 6	$2r \geq d - r_G$	+1	+1
Figure 7(a)	none	+1	+1
Figure 7(b)	$2r \leq d + r_G$	-1	+1
Figure 7(c)	none	+1	-1
Figure 7(d)	$2r \leq r_G - d$	-1	-1

Tab. 1: Identifying cases and subcases.

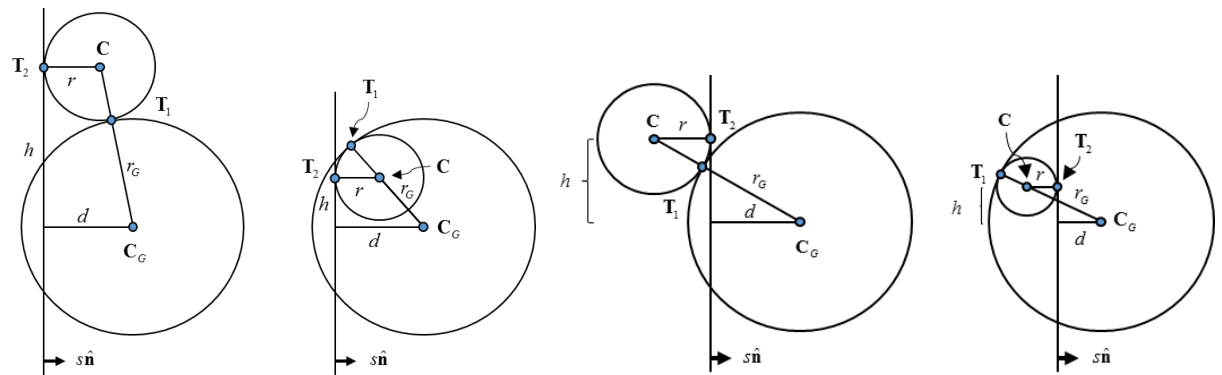


Fig. 7: Intersecting cases: (a) $s_1 > 0, s_2 > 0$, (b) $s_1 < 0, s_2 > 0$, (c) $s_1 > 0, s_2 < 0$, (d) $s_1 < 0, s_2 < 0$.

The cases and subcases are summarized in Table 1 and illustrated in Figure 6 and Figure 7. Compute h , the center \mathbf{C} , the tangent vector to the center path arc $\mathbf{T} = d\mathbf{C}/dr$, and the points of tangency \mathbf{T}_1 and \mathbf{T}_2 , as follows: $h = \sqrt{r_G^2 + s_1 2rr_G - d^2 + s_2 2rd}$, $\mathbf{C} = \mathbf{C}_G + s_{ul} h s\hat{\mathbf{n}}^\perp + s_2 (r - s_2 d) s\hat{\mathbf{n}}$, $h' = (s_1 r_G + s_2 d) / \sqrt{r_G^2 + s_1 2rr_G - d^2 + s_2 2rd}$, $\mathbf{T} = s_{ul} h' s\hat{\mathbf{n}}^\perp + s_2 s\hat{\mathbf{n}}$, $\mathbf{T}_1 = (r_G / (r_G + s_1 r)) \mathbf{C} + (s_1 r / (r_G + s_1 r)) \mathbf{C}_G$, and $\mathbf{T}_2 = \mathbf{C} - s_2 r s\hat{\mathbf{n}}$.

Circle Tangent to Two Given Circles

The possible locations of the constructed circles considered are shown in Figure 8. ([2] has a complete list of cases and the method works on all of them. For brevity, only those in Figure 8 are treated here.)

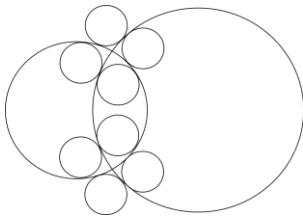


Fig. 8: Cases considered.

Inside or outside circle #1	s_1	$\text{signum}(\ \mathbf{P}_{pick} - \mathbf{C}_1\ - r_1)$
Inside or outside circle #2	s_2	$\text{signum}(\ \mathbf{P}_{pick} - \mathbf{C}_2\ - r_2)$
Above or below center line	s_{ul}	$\text{signum}((\mathbf{P}_{pick} - \mathbf{C}_1) \cdot \hat{\mathbf{u}}^\perp)$
Overlapping circles	s_{dr}	$\text{signum}_+(d - r_1 - r_2)$
One circle inside other	s_d	$\text{signum}_-(d + \min(r_1, r_2) - \max(r_1, r_2))$

Tab. 2: Definitions of case selectors.

Let r be the given radius and let the centers of the given circles be \mathbf{C}_1 and \mathbf{C}_2 with radii r_1 and r_2 , respectively. The distance between the centers of the given circles is $d = \|\mathbf{C}_2 - \mathbf{C}_1\|$ and $\hat{\mathbf{u}}$ is a unit vector in the direction from \mathbf{C}_1 to \mathbf{C}_2 . The quantities s_1, s_2, s_{ul}, s_{dr} and s_d , as defined in Table 2, are used to manage the cases. (For the packing problem, s_1, s_2 and s_{ul} are specified directly.)

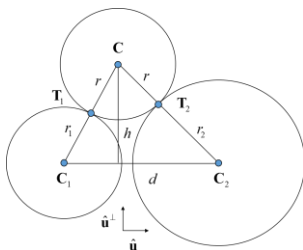


Fig. 9: External case.

Cases	Restrictions	s_1	s_2	s_{dr}	s_d
Figure 9	$2r \geq d - r_1 - r_2$	+1	+1	+1	+1
Figure 10(a)	$2r \leq r_1 + r_2 - d$	-1	-1	-1	+1
Figure 10(b)	$2r \leq r_1 - r_2 + d$	-1	+1	-1	+1
Figure 10(c)	$2r \leq -r_1 + r_2 + d$	+1	-1	-1	+1
Figure 11(a)	$r_2 - r_1 - d \leq 2r \leq r_2 - r_1 + d$	+1	-1	-1	-1
Figure 11(b)	$r_1 - r_2 - d \leq 2r \leq r_1 - r_2 + d$	-1	+1	-1	-1

Tab. 3: Summary of cases.

In Figure 9, the circles do not overlap. If $r_1 < r_2$ then $\hat{\mathbf{u}} \cdot (\mathbf{C} - \mathbf{C}_1)$ can be positive or negative depending on the value of r . Compute the sign as $s_b = \text{signum}((r_1 + r)^2 + d^2 - (r_2 + r)^2)$ (Law of Cosines). The sign change occurs in other cases, but with differing signs in s_b . The formulas for this case, including s_b , also work in a related case where the circles overlap but one is not contained in the other, i.e. $s_{dr} < 0$ and $s_d > 0$, and the constructed circle is outside the given circles, i.e. $s_1 > 0$ and $s_2 > 0$.

The other cases where the given circles overlap but one is not contained in the other, are identified as $s_{dr} < 0$ and $s_d > 0$. The cases are shown in Figures 10(a), 10(b) and 10(c). Note also Table 3.

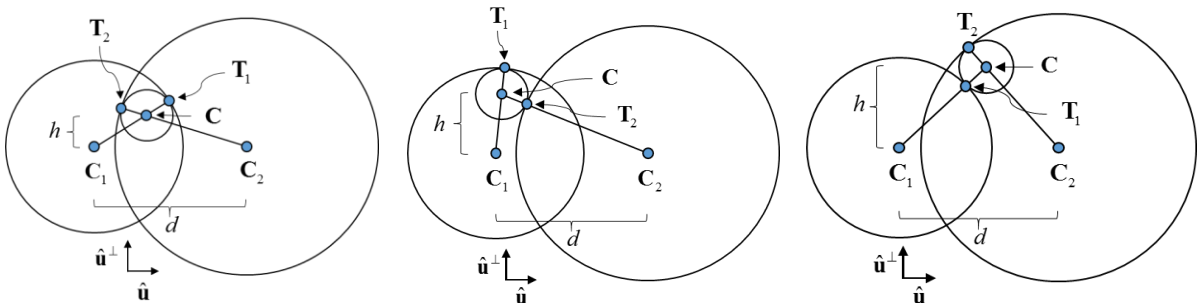


Fig. 10: Interior cases: (a) inside both, (b) inside 1, outside 2, (c) outside 1, inside 2.

A final case is one given circle inside the other, Figure 11. Prior to examining the subcase in Figure 11(b), combine all of the previous cases and subcases, including Figure 11(a).

The calculations are the following. Compute, h , the height of the triangle formed by C , C_1 and C_2 and compute $s_b = \text{signum}\left((r_1 + s_1 r)^2 + d^2 - (r_2 + s_2 r)^2\right)$. Use these to compute center of the circle, C , the tangent vector to the center path, $\mathbf{T} = d\mathbf{C}/dr$, and the points of tangency, \mathbf{T}_1 and \mathbf{T}_2 , as follows:

$$h = \sqrt{\left(d^2 - (r_1 - s_1 s_2 r_2)^2\right)\left((r_1 + s_1 s_2 r_2 + s_1 2r)^2 - d^2\right)} / (2d), \quad \mathbf{C} = \mathbf{C}_1 + s_b \sqrt{(r + r_1)^2 - h^2} \hat{\mathbf{u}} + s_{ul} h \hat{\mathbf{u}}^\perp,$$

$$h' = s_1 \left(d^2 - (r_1 - s_1 s_2 r_2)^2\right) (r_1 + s_1 s_2 r_2 + 2s_1 r) / (2d^2 h), \quad \mathbf{T} = s_b \left((r + r_1 - hh') / \sqrt{(r + r_1)^2 - h^2}\right) \hat{\mathbf{u}} + s_{ul} h' \hat{\mathbf{u}}^\perp,$$

$$\mathbf{T}_1 = (s_1 r / (r_1 + s_1 r)) \mathbf{C}_1 + (r_1 / (r_1 + s_1 r)) \mathbf{C}, \text{ and } \mathbf{T}_2 = (s_2 r / (r_2 + s_2 r)) \mathbf{C}_2 + (r_2 / (r_2 + s_2 r)) \mathbf{C}$$

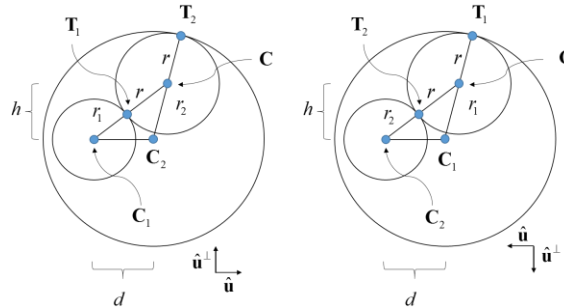


Fig. 11: Other enclosing cases: (a) 1 inside 2, (b) 2 inside 1.

The formula for Figure 11(b) reverses the roles of 1 and 2 in Figure 11(a). With h and h' as above and with $s_b = \text{signum}\left((r_2 + r)^2 + d^2 - (r_1 - r)^2\right)$, compute the center, tangent vector and tangent points as follows:

$$\mathbf{C} = \mathbf{C}_2 - s_b \sqrt{(r_2 + r)^2 - h^2} \hat{\mathbf{u}} + s_{ul} h \hat{\mathbf{u}}^\perp, \quad \mathbf{T} = -s_b \left((r + r_2 - hh') / \sqrt{(r + r_2)^2 - h^2}\right) \hat{\mathbf{u}} + s_{ul} h' \hat{\mathbf{u}}^\perp,$$

$$\mathbf{T}_1 = (-r / (r_1 - r)) \mathbf{C}_1 + (r_1 / (r_1 - r)) \mathbf{C}, \text{ and } \mathbf{T}_2 = (r / (r_2 + r)) \mathbf{C}_2 + (r_2 / (r_2 + r)) \mathbf{C}.$$

Summary:

The method gives simple and elegant formulas for the fittings and for the center path arc and its tangent vector. The authors have done derivations of common CAD constructions involving lines and circles (a total of 21, including the constructions presented herein), which give simple formulas that are easily coded and easily manipulated analytically. Although not relevant to the current analysis, the formulas do allow the possibility of determining the sensitivity of the solutions to various variables.

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