Title:
A New Log-aesthetic Space Curve Based on Similarity Geometry

Authors:
Kenjiro T. Miura, miura.kenjiro@shizuoka.ac.jp, Shizuoka University
Sho Suzuki, suzuki.sho.14a@shizuoka.ac.jp, Shizuoka University
R.U. Gobithaasan, gr@umt.edu.my, Universiti Malaysia Terengganu
Shin Usuki, usuki@shizuoka.ac.jp, Shizuoka University

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Introduction:
In similarity geometry, we identify figures overlapped by similarity transformations in addition to the Euclidean congruent transformations. We would like to review the theory of plane and space curves in similarity geometry as well as Euclidean geometry.

Similarity Geometry In 2D
In this section, we derive similarity Frenet frame to introduce the definition of similarity curvature and show its role in similarity geometry [2]. Since we know that the arc length s may vary, thus the representation of plane curves is in the form of direction angle θ parameterized which is invariant by scaling. First, let a plane curve $\boldsymbol{C}(s)$ is given as a function of its arc length by

$$\boldsymbol{C}(s) = (x(s), y(s)) \tag{1}$$

and its Frenet frame $F(s) = (T(s), N(s))$. We assume the curve is not a straight line and the direction angle θ is defined by

$$\theta = \int_0^s \kappa(s)\, ds. \tag{2}$$

Next, let tangent vector $T^{Sim}(\theta)$ as follows to define the Frenet frame in similarity geometry,

$$T^{Sim}(\theta) \equiv \frac{d\boldsymbol{C}}{d\theta}(\theta). \tag{3}$$

Thus, we may simplify as

$$T^{Sim}(\theta) = \frac{dC}{ds}\frac{ds}{d\theta} = \frac{1}{\kappa(s)}T(s) \tag{4}$$

where $T(s)$ is the first derivative of $C(s)$ with respect to s and it is a unit tangent vector of the curve. Let $N^{Sim}(\theta)$ be

$$N^{Sim}(\theta) = \frac{1}{\kappa(s)}N(s) \tag{5}$$

Since $\det\left(T^{Sim}, N^{Sim}\right) = 1/\kappa^2$, hence $F^{Sim}(\theta) = (T^{Sim}(\theta), N^{Sim}(\theta))$ has a value in

$$CO^{+}(2) = \{X \in CO(2) | \det X > 0\} \tag{6}$$

where $CO^{+}(2)$ is a set of 2×2 real matrix A such that $AA^{T} = cE$ for an arbitrary constant c. Here A^{T} denotes a transpose of matrix A and E does a unit matrix. The derivatives of $T^{Sim}(\theta)$ and $N^{Sim}(\theta)$ are given by

$$\frac{d}{d\theta}T^{Sim}(\theta) = -\frac{\kappa_s(s)}{\kappa(s)^2}T^{Sim}(\theta) + N^{Sim}(\theta) \tag{7}$$

$$\frac{d}{d\theta}N^{Sim}(\theta) = -\frac{\kappa_s(s)}{\kappa(s)^2}N^{Sim}(\theta) - T^{Sim}(\theta) \tag{8}$$

From equation (7) and (8), we define

$$S(\theta) = \frac{\kappa_s(s)}{\kappa(s)^2} \tag{9}$$

Equation (9) is an invariant in similarity geometry and it is denoted as similarity curvature. Therefore, $F^{Sim}(\theta)$ satisfies the following differential equation:

$$\frac{d}{d\theta}F^{Sim}(\theta) = F^{Sim}(\theta)\begin{pmatrix} -S(\theta) & -1 \\ 1 & -S(\theta) \end{pmatrix} \tag{10}$$

.

The above equation is called the formula of Frenet frame in similarity geometry.

Similarity Geometry In 3D

Let $\boldsymbol{C}(s)$ be a smooth space curve parametrized by an arclength s from the start point of the curve. The unit tangent vector of the curve $\boldsymbol{T}(s)$ is given by a vector $d\boldsymbol{C}(s)/ds$ and the unit normal vector $\boldsymbol{N}(s)$ is defined as a unit vector whose direction is in the same direction as the derivative of $\boldsymbol{T}(s)$ with respect to the arc length. The unit bi-normal vector $\boldsymbol{B}(s)$ is given by the cross product of $\boldsymbol{T}(s)$ and $\boldsymbol{N}(s)$. The triplet of the unit vector [$\boldsymbol{T}(s)$, $\boldsymbol{N}(s)$, $\boldsymbol{B}(s)$] set in 3×3 matrix is called a frame. Differentiating the frame, we obtain the Frenet-Serret formula as follows [1]:

$$\left[\frac{d\boldsymbol{T}(s)}{ds}, \frac{d\,\boldsymbol{N}(s)}{ds}, \frac{d\,\boldsymbol{N}(s)}{ds}\right] = [\boldsymbol{T}(s), \boldsymbol{N}(s), \boldsymbol{B}(s)]\begin{bmatrix} 0 & -\kappa(s) & 0 \\ \kappa(s) & 0 & -\tau(s) \\ 0 & \tau(s) & 0 \end{bmatrix} \tag{11}$$

In the similarity geometry since an arc length is not invariant by similarity transformations, we parametrize a curve as $\boldsymbol{C}(\theta)$ where θ is defined by $d\theta/ds = \kappa(s)$. For the plane curve, θ corresponds to the relative direction angle θ from the start point of the curve. θ is an invariant parameter in similarity geometry.

Let $\boldsymbol{T}^{sim}(\theta) = d\boldsymbol{T}(s)/d\theta$ be the similarity tangent vector. Then

$$\boldsymbol{T}^{sim}(\theta) = \frac{d\boldsymbol{C}}{d\theta} = \frac{ds}{d\theta}\frac{d\boldsymbol{C}}{ds} = \frac{1}{\kappa(s)}\boldsymbol{T}(s) = \rho(s)\boldsymbol{T}(s) \tag{12}$$

Similarly, we define $\boldsymbol{N}^{sim}(\theta) = \boldsymbol{N}(s)/\kappa(s)$ and $\boldsymbol{B}^{sim}(\theta) = \boldsymbol{B}(s)/\kappa(s)$ as the similarity tangent and bi-normal vectors, respectively.

Differentiating the similarity frame [$\boldsymbol{T}^{sim}(\theta)$, $\boldsymbol{N}^{sim}(\theta)$, $\boldsymbol{B}^{sim}(\theta)$], we obtain the similarity Frenet-Serret formula as follows:

$$\left[\frac{d\boldsymbol{T}^{sim}(\theta)}{d\theta}, \frac{d\,\boldsymbol{N}^{sim}(\theta)}{d\theta}, \frac{d\,\boldsymbol{B}^{sim}(\theta)}{d\theta}\right] = \left[\boldsymbol{T}^{sim}(\theta), \boldsymbol{N}^{sim}(\theta), \boldsymbol{B}^{sim}(\theta)\right]\begin{bmatrix} -\tilde{\kappa} & -1 & 0 \\ 1 & -\tilde{\kappa} & -\tilde{\tau} \\ 0 & \tilde{\tau} & -\tilde{\kappa} \end{bmatrix} \tag{13}$$

where

$$\tilde{\kappa} = \frac{1}{\kappa^2}\frac{d\kappa}{ds} = -\frac{d\rho}{ds} = -\frac{1}{\rho}\frac{d\rho}{d\theta} \tag{14}$$

$$\tilde{\tau} = \frac{\tau}{\kappa} = \rho\tau \tag{15}$$

$\tilde{\kappa}$ are called similarity curvature and we call $\tilde{\tau}$ as similarity torsion.

Log-aesthetic Curve:
The following Theorem was proved by Sato and Shimizu [6] and stated as a theorem by Inoguchi [2]:
Theorem 1. Log-aesthetic curves are characterized as plane curves whose reciprocal similarity curvature $1/\tilde{\kappa}(\theta)$ is a linear function of θ.

Miura [3] has mathematically formulated a family of curves which has strictly linear logarithmic curvature graph with the slope α and derived a general formula for the radius of curvature as follows (here ρ is normalized to be 1 at $s = 0$):

$$\rho(s) = \begin{cases} e^{\lambda s} & \alpha = 0, \\ (\lambda\alpha s + 1)^{\frac{1}{\alpha}} & \alpha \neq 0. \end{cases} \tag{16}$$

$$\theta(s) = \begin{cases} \frac{1-e^{-\lambda s}}{\lambda} & \alpha = 0, \\ \frac{\log(\lambda s+1)}{\lambda} & \alpha = 1, \\ \frac{(\lambda\alpha s+1)^{\frac{\alpha-1}{\alpha}}-1}{\lambda(\alpha-1)} & \text{otherwise.} \end{cases} \tag{17}$$

Log-aesthetic Space Curve
The radius of torsion μ_{LASC} of the log-aesthetic space curve is defined by

$$\mu_{LASC} = \begin{cases} ce^{ds} & \beta = 0, \\ (cs + d)^{\frac{1}{\beta}} & \beta \neq 0 \end{cases} \tag{18}$$

in order to let it have self-affinity [4]. Here c and d are arbitrary constants. Hence the reciprocal similarity torsion $\tilde{\mu}_{LASC}$_LASC (or the similarity radius of torsion [5] of LAS is given by

$$\tilde{\mu}_{LASC} = \frac{1}{\tilde{\tau}_{LASC}} = \kappa_{LASC}\mu_{LASC} \tag{19}$$

where

$$\kappa_{LASC} = \begin{cases} e^{-\lambda s} & \alpha = 1, \\ (\lambda(\alpha-1)\theta + 1)^{\frac{1}{1-\alpha}} & \alpha \neq 1. \end{cases} \tag{20}$$

μ_{LASC} is relatively very complicated depending on α and β values. When $\beta = 0$,

$$\mu_{LASC} = \begin{cases} c(1-\lambda\theta)^{-\frac{d}{\lambda}} & \alpha = 0, \\ ce^{\frac{d}{\lambda}(e^{\theta}-1)} & \alpha = 1, \\ ce^{\frac{d}{\lambda\alpha}\{(1+(\alpha-1)\lambda\theta-1)^{\frac{\alpha}{\alpha-1}}-1\}} & \text{otherwise.} \end{cases} \tag{21}$$

Otherwise, i.e. when $\beta \neq 0$,

$$\mu_{LASC} = \begin{cases} (-\frac{c}{\lambda}\log(1-\lambda\theta) + d)^{\frac{1}{\beta}} & \alpha = 0, \\ (\frac{c}{\lambda}(e^{\lambda\theta} - 1) + d)^{\frac{1}{\beta}} & \alpha = 1, \\ \{\frac{c}{\lambda\alpha}\left((1 + (\alpha-1)\lambda\theta)^{\frac{\alpha}{\alpha-1}} - 1\right) + d\}^{\frac{1}{\beta}} & \text{otherwise.} \end{cases} \tag{22}$$

Even though the similarity radius of curvature of LAC is given by a linear function of θ, the similarity radius of torsion of LASC is rather complicated.

New Extension
In this section, we define a new space curve by assuming the fact stated in **Theorem 1.** and that the reciprocal similarity torsion is given by a linear function of θ. We can determine the radius of torsion $\mu = 1/\tau$ instead of the torsion τ of the newly defined space curve for simplicity of the derived expression. Assuming $1/\tilde{\tau} = \tilde{\mu} = a\theta + b$ in Eqn. (19) μ is given by

$$\mu = \rho(a\theta + b) \tag{23}$$

where a and b are arbitrary constants. From Eqn. (17) and (18), when $\alpha = 0$,

$$\mu = e^{\lambda s}\left(a\frac{1-e^{-\lambda s}}{\lambda} + b\right) = \left(\frac{a}{\lambda} + b\right)e^{\lambda s} - \frac{a}{\lambda} \tag{24}$$

When $\alpha = 1$,

$$\mu = (\lambda s + 1)(\frac{a}{\lambda}\log(\lambda s + 1) + b) \tag{25}$$

Otherwise,

$$\begin{aligned}\mu &= (\lambda\alpha s + 1)^{\frac{1}{\alpha}}\left(a\frac{(\lambda\alpha s + 1) - 1}{\lambda(\alpha - 1)} + b\right) \\ &= \frac{a}{\lambda(\alpha-1)}(\lambda\alpha s + 1) + (b - \frac{a}{\lambda(\alpha-1)})(\lambda\alpha s + 1)^{\frac{1}{\alpha}} \\ &= a_0(\lambda\alpha s + 1) + b_0(\lambda\alpha s + 1)^{\frac{1}{\alpha}}\end{aligned} \tag{26}$$

where $a_0 = a/\lambda(\alpha - 1)$ and $b_0 = b - a/\lambda(\alpha - 1)$.

As in Eqn. (19) the original definition has a parameter β and its properties are not well understood, especially its impressions although the parameter α has been well studied. Our new definition looks similar with the original one if we assume that $\beta = \alpha$, but still somewhat different.

Figure 1 shows a comparison of LASCs defined by our new formulation and defined traditionally and Figure 2 shows graphs of their curvature and torsion distributions. Both values of α and β are equal to -0.5 and their curvatures are given by

$$\kappa(s) = 0.04s^2 \tag{27}$$

The torsion τ_n of the newly defined LASC is

$$\tau_n(s) = \frac{0.06s^2}{0.16\theta+0.81} \approx \frac{0.06s^2}{0.213s^{3+0.81}} \tag{28}$$

The torsion τ_t of the traditionally defined LASC is

$$\tau_n(s) = 0.06s^2 \tag{29}$$

Their curvature distributions are the same and their torsions start from 0 at the start points (the coordinate origin) and gradually increase.

Conclusions:

We have derived a new space curve formula based on the log-aesthetic plane curve with the similarity geometry, in which we have utilized that the slope α of the logarithmic curvature graph automatically determines the slope β of the logarithmic torsion graph. We have also presented a generation method of the log-aesthetic space curve in case where it is a log-aesthetic magnetic curve.

In the future work we will extend our generation method of the log-aesthetic curve including $a \neq 0$ case, improve its efficiency and find out its drawable region where the curve can be generated. We will investigate more properties of similarity torsion, which is an invariant in similarity geometry and its effects on aesthetic design and evaluate the quality of space curves defined by similarity torsion.

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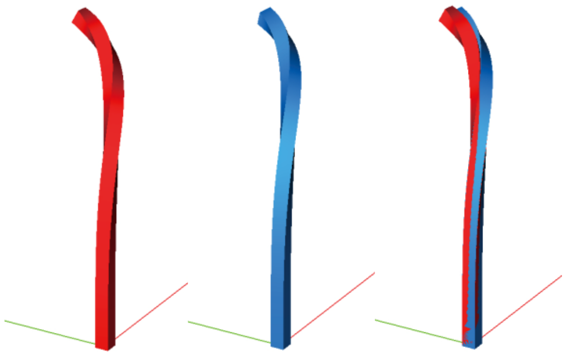

Fig. 1: From left to right, LASC by our new definition, traditional LASC, both rendered at the same place.

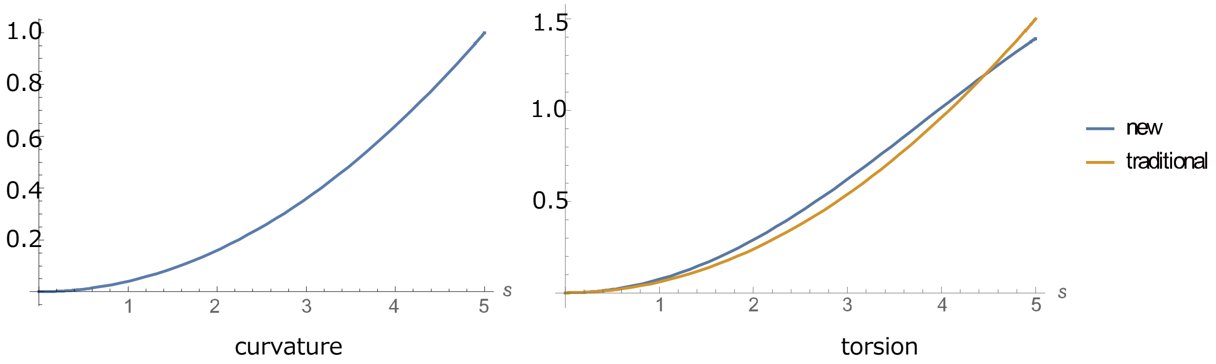

Fig. 2: Curvature and torsion distributions.

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