Title:
**Time Optimal Driving on Curvilinear Path with Kinematic Constraints**

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Introduction:
In work such as parts transport and processing, the parts or the effectors of a robot are often driven on a specified curvilinear path. Also, a car or a train often runs on a curved road or railway. When moving an object on a curved path, the force acting on the object includes not only the component in tangential direction but also the component in normal direction. Since the force acting on moving object is influenced by curvature and torsion of the path, they must be taken into account. How is an object driven through a curvilinear path as quickly as possible while limiting the force acting on it? It is the subject to be studied in this paper.

In the following sections, we discuss the necessary constraints, formulating curvilinear motion, algorithm, a calculated example and the verification by using an actual manipulator.

Previous Studies:
So far, cam curves have been often used for path driving [13]. These curves are formulated by some elementary functions, so they are easy to be used. However, these curves were originally created for linear motion, not for curvilinear path. Its driving velocity does not take into account the change in curvature of the path and it takes more driving time than necessary.

As a study of the shortest time driving on the curvilinear path, Bobrow et al. suggested a method by considering limitations of angular velocity and torque of actuators [1]. They expressed the dynamics of the driving system with curve length of the path. Then, the minimum time driving pattern is created by maximally accelerating and maximally decelerating segments. By this mean, the resultant driving pattern guarantees shortest time driving. However, the acceleration in acceleration-deceleration switching position is not continuous. This driving method is so-called bang-bang control, and is evidently not a smooth driving.

In order to overcome the disadvantages of the bang-bang control described above, some improving studies were carried out [3-4],[6-7],[10-12],[16]. In addition to limiting the angular velocity and the maximum torque of actuators, the limitation on torque's time derivative was also considered. However, in practical application, one wishes to limit the forces acted on an object moving on the path rather than its actuator's torque. These two subjects are actually not equivalent. For example, when rotating a manipulator's arm fixed on the vertical shaft of a motor with high speed, an object fixing on the arm top moves on a circular orbit. In this case, a strong centrifugal force will be acted on it (in radial direction), but this force is absorbed by the bearing of rotating shaft and is not reflected on rotational torque of the motor. A car passing through a curvilinear road at high speed without considering this centrifugal force may face dangers of falling or side-slipping.

In this research, the focus is put not on torque of the actuators but on object moving on its path with given kinematic constraints such as velocity, acceleration and jerk. Up to now, this approach has not been well studied.

## Constraints and Formulation:

*Kinematic Constraints*

In curvilinear motion, velocity, acceleration and jerk of the moving object are vectors. We take maximum absolute values of those vectors as the constraints. The velocity need to be limited for safety reasons. Limitation of maximum acceleration is important because it is proportional to the force acted on the moving object. By limiting maximum jerk, the changing rate of acceleration is restricted and smooth movement can be realized.

In actual operation, the constraints of velocity, acceleration and jerk are decided by the purpose of application. For example, when moving a fragile object such as a cake, both acceleration and jerk ought to be lowly limited. By setting the constraints suitably, it is possible to achieve time optimal driving while reducing the moving vibration.

*Formulation of Curvilinear Motion*

For brevity, we consider the moving object as a mass point. A curvilinear path is shown in Fig. 1. Here, $s$ is the curve length from path start point, $\boldsymbol{h}$ represents total path length and $\boldsymbol{r}(s)$ represents position vector of the moving mass point. The definitions of velocity $\boldsymbol{v}$, acceleration $\boldsymbol{a}$ and jerk $\boldsymbol{j}$ are given in equation (1).

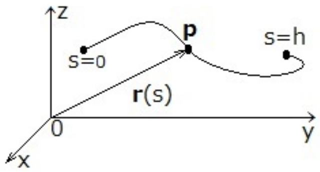


Fig. 1: A particle moving on curvilinear path.

$$\begin{cases} \boldsymbol{v} = \dot{\boldsymbol{r}}(s) = d\boldsymbol{r}(s)/dt \\ \boldsymbol{a} = \ddot{\boldsymbol{r}}(s) = d^2\boldsymbol{r}(s)/d\boldsymbol{t}^2 \\ \boldsymbol{j} = \dddot{\boldsymbol{r}}(s) = d^3\boldsymbol{r}(s)/d\boldsymbol{t}^3 \end{cases} \tag{1}$$

$$\begin{cases} d\boldsymbol{T}/ds = \kappa\boldsymbol{N} \\ d\boldsymbol{N}/ds = -\kappa\boldsymbol{T} + \tau\boldsymbol{B} \\ d\boldsymbol{B}/ds = -\tau\boldsymbol{N} \end{cases} \tag{2}$$

$$\begin{cases} \boldsymbol{v} = \dot{s}\boldsymbol{T} \\ \boldsymbol{a} = \ddot{s}\boldsymbol{T} + \dot{s}^2\kappa\boldsymbol{N} \\ \boldsymbol{j} = (\dddot{s} - \dot{s}^3\kappa^2)\boldsymbol{T} + \ (3\dot{s}\ddot{s}\kappa + \dot{s}^3\kappa')\boldsymbol{N} + \dot{s}^3\kappa\tau\boldsymbol{B} \end{cases} \tag{3}$$

At an arbitrary point $\boldsymbol{P}$ on the curvilinear path, we express the corresponding curvature as $\kappa$ and the corresponding torsion as $\tau$. The Frenet-Serret formulas [9] are shown in Eqn. (2). Here, ***T, N*** and ***B*** are unit vectors in tangential, normal and bi-normal directions respectively. Notice $d/dt = (ds/dt)\ d/ds$ and $dr/ds = \boldsymbol{T}$, we can combine Eqn. (1) and (2) to obtain Eqn. (3). Here, $\dot{s}$, $\ddot{s}$ and $\dddot{s}$ are the first, second and third order derivatives of $s$ with respect to time, $\kappa'$ is derivative of the curvature to curve length. In Eqn. (3), $\dot{s}$ represents the velocity (tangential direction), $\ddot{s}$ and $\dot{s}^2\kappa$ represent tangential and normal acceleration component, $\dddot{s} - \dot{s}^3\kappa^2$, $3\dot{s}\ddot{s}\kappa + \dot{s}^3\kappa'$ and $\dot{s}^3\kappa\tau$ represent the tangential, normal and bi-normal jerk components respectively.

*Formulating the Problem*

Let $V$, $A$ and $J$ represent the constraints (the maximum absolute values of $\boldsymbol{v}$, $\boldsymbol{a}$ and $\boldsymbol{j}$ ), $h$ represents the total path length, $t_f$ represents the final moving time. Our optimizing problem can be expressed as Eqn. (4). The limitations of $\dot{s}$, $\ddot{s}$ and $\dddot{s}$ are determined in Eqn. (5).

$$\begin{cases} ds/dt = \dot{s} \\ d^2s/dt^2 = \ddot{s} \\ d^3s/dt^3 = \dddot{s} \\ 0 \le s \le h \\ 0 \le \dot{s} \le V \\ \ddot{s}^2 + \dot{s}^4\kappa^2 \le A^2 \\ (\dddot{s} - \dot{s}^3\kappa^2)^2 + (3\dot{s}\ddot{s}\kappa + \dot{s}^3\kappa')^2 + \dot{s}^6\kappa^2\tau^2 \le J^2 \\ s(0) = 0,\ s(t_f) = h \\ \dot{s}(0) = 0,\ \dot{s}(t_f) = 0 \\ \ddot{s}(0) = 0,\ \ddot{s}(t_f) = 0 \\ minimize[t_f = \int_0^h (1/\dot{s})ds] \end{cases} \tag{4}$$

$$\begin{cases} \dot{s}_H = \min\{V, (A/\kappa)^{1/2}\} \\ \ddot{s}_H = (A^2 - \dot{s}^4\kappa^2)^{1/2} \\ \ddot{s}_L = -(A^2 - \dot{s}^4\kappa^2)^{1/2} \\ \dddot{s}_H = \dot{s}^3\kappa^2 + (J^2 - (3\dot{s}\ddot{s}\kappa + \dot{s}^3\kappa')^2 - \dot{s}^6\kappa^2\tau^2)^{1/2} \\ \dddot{s}_L = \dot{s}^3\kappa^2 - (J^2 - (3\dot{s}\ddot{s}\kappa + \dot{s}^3\kappa')^2 - \dot{s}^6\kappa^2\tau^2)^{1/2} \end{cases} \tag{5}$$

Notice that the acceleration and jerk of a moving object are $\boldsymbol{a}$ and $\boldsymbol{j}$, not $\ddot{s}$ and $\dddot{s}$. The latter ones are simply used as convenient variables for finding the time optimal driving pattern. Actually, $\ddot{s}$ is the tangential component of $\boldsymbol{a}$, but $\dddot{s}$ is not even the tangential component of $\boldsymbol{j}$. The tangential component of $\boldsymbol{j}$ is $\dddot{s} - \dot{s}^3\kappa^2$ indeed.

<u>Algorithm:</u>

Eqn. (4) are differential equations with assigned boundary condition and nonlinear constraints on state variables. The solution to this kind of problem has not been well established [2],[8],[14-15]. The algorithm proposed here can be considered as the jerk extension of Bobrow et al. The algorithm uses bisection method. Though this is not a fast algorithm, but it can give accurate results and it is reliable. First, we define **AME** and **BME** driving pattern as follows.

**AME:** Accelerating with Maximum Effort by using $\dddot{s}_H$ and $\ddot{s}_H$.

**BME:** Braking to stop with Maximum Effort by using $\dddot{s}_L$ and $\ddot{s}_L$. At the braking end, both $\dot{s} = 0$ and $\ddot{s} = 0$ must be satisfied (It means that $\ddot{s}$ must be bring to zero by using $\dddot{s}_H$ when reaching stop).

**Calculating Procedure**

1. Set $s_1 = 0$, $\dot{s}_1 = 0$ and $\ddot{s}_1 = 0$.

2. From $s_1$, $\dot{s}_1$ and $\ddot{s}_1$ state, calculate **AME** until it contradicts the given constraints. Denote the final state as $s_2$, $\dot{s}_2$ and $\ddot{s}_2$.

3. For $[s_1,\ s_2]$, using bisection method to calculate the optimal point which switches from accelerating to braking. This procedure is an iterative calculating of **AME** and **BME**.

4. If $s$ at the end of Step.3 just reaches final position $s = h$, all calculating is finished. Otherwise, move one mini step along the calculated **BME**. Set the new state to $s_1$, $\dot{s}_1$ and $\ddot{s}_1$, then repeat Step.2.

The iterative calculation of **AME** and **BME** is the core of this algorithm. Notice $\ddot{s}$ must be continuous because of jerk constraint.

<u>Calculation result:</u>

As calculating examples, the time optimal driving pattern on a s-shape path are shown in Fig. 2. The path shape, curvature, torsion and the curvature's derivative are plotted with respect to curvilinear path length. The speed, acceleration and jerk of the calculated time optimal driving pattern are plotted with respect to time. $\dot{s}(t)$, $\ddot{s}(t)$ and $\dddot{s}(t)$ are drawn with thick lines and the others are drawn with thin lines. The calculation conditions are listed as follows.

1) Constraints:
   $V = 1000$ mm/s.
   $A = 1000$ mm/s$^2$.
   $J = 5000$ mm/s$^3$.
2) Path length $h = 1000$ mm.
3) Mini time step adopted for calculation: $\Delta t = 1$ ms.

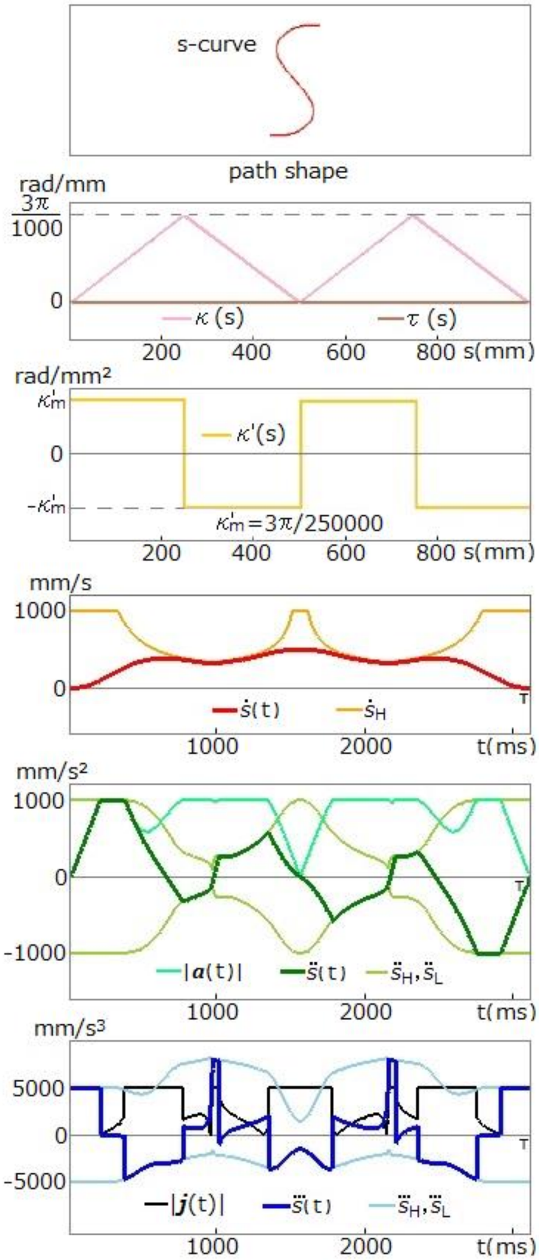


Fig. 2: Minimum time driving patterns on s-curve of $h = 1000$ mm.

On the s-shape path, the curvature is zero at the starting point, the midpoint and the end point. The path has a segmental linear curvature distribution. Since the curvature changes along the path, the limitations of $\dot{s}(t)$, $\ddot{s}(t)$ and $\dddot{s}(t)$ possess complicated shapes. The maximum curvature is $\kappa_{max} = 3\pi/1000$ rad/mm. Maximum value of calculated velocity is $\dot{s}_{max} = 502.713$ mm/s and the required driving time is T = 3129 ms.

Verification:

The actual verification is shown in Fig. 3 and Fig. 4. This is an experiment of tracking s-shape path using a manipulator with orthogonally driving arms. The driving data to each motor is sending via EtherCAT which transmits data at high speed with 1 ms cycle [5]. Since the movable range of this manipulator cannot draw s-shape of 1000 mm length, a half scale condition is adopted as following.

1) Constraints:
   $V = 500$ mm/s.
   $A = 500$ mm/s$^2$.
   $J = 2500$ mm/s$^3$.
2) Path length $h = 500$ mm.
3) Mini time step adopted for calculation : $\Delta t = 1$ ms.

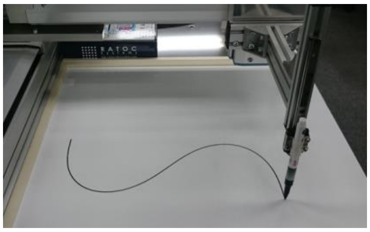

Fig. 3 Plot s-curve by a manipulator

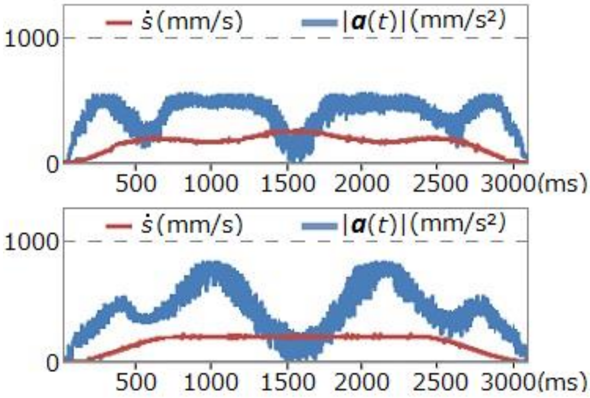


Fig. 4: The driving result on s-shape path using actual actuator.

Fig. 4 is the measured result (created with the angle sensor on motor shaft) by using both time optimal driving pattern and traditional cam driving pattern (the modified constant velocity CAM curve). With the same driving time, the maximum acceleration when using the time optimal pattern is lower than that of using the modified constant velocity CAM pattern. This means smoother curvilinear movement is realized.

Conclusions:
In this paper, an algorithm for calculating time optimal driving pattern on a curvilinear path is obtained. The driving pattern satisfies constraints of velocity, acceleration and jerk. By applying restraint of jerk, noise, vibration and mechanical fatigue can be reduced and smooth driving is obtained.

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