Title:
On a Blossom-based Approach for Interpolating Non-iso-Parametric Curves by B-Spline Surfaces

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Introduction:
Curve interpolation is a problem that has frequently been visited by many authors [2, 12, 13]. However, the proposed solutions rely in a fundamental way on the assumption that the interpolated curve is iso-parametric along one or the other of the parameters of the interpolating surface, in so much that, as soon as this assumption is removed, the level of difficulty of the interpolation problem rises enormously.

Ferguson and Grandine [4] are perhaps the first to aim for the construction of B-spline surfaces interpolating non-iso-parametric curves. However, to our knowledge, there was never a follow up to that paper. Most of related research may rather be found in the area of curves on surfaces [16] using blossoming techniques [15] and also in the area of interpolation of arbitrary networks of curves [9].

However, the work of Hu&Sun [6] is directly relevant to the research reported here. Especially so, since the technique proposed here for the solution of the interpolation of non-iso-parametric curves is a slight adaptation of the one reported in that paper and used there for trimmed surface matching.

Main Idea:
We first show how the basic definition of B-spline curves and surfaces can naturally lead to the notion of B-spline polygonal complexes [1, 11] and therefore to curve interpolation. Furthermore, we also shed light on the role iso-paramatricity plays in achieving curve interpolation directly and at very little cost.

B-Spline Curves
Given a sequence of control points $[p_0, p_1, ..., p_m]$, a non-decreasing sequence of knots $[t_0, t_1, ..., t_n]$, and a parameter $t \in [t_p..t_{m+1}]$, a B-spline curve of degree p (such that $m = n - p - 1$) is defined as follows:

$$C(t) \equiv \sum_{i=0}^{m} N_i^p(t)\, p_i \tag{1}$$

where N_i^p is the i^{th} B-spline basis function of degree p.

In general, at most $p+1$ control points affect the curve $C(t)$ at parameter t. These have the consecutive indices from $i-p$ to i, where $t_i \le t < t_{i+1}$.

Thus, for example, in the cubic case (i.e., when $p=3$), for any parameter w ($t_k \le w < t_{k+1}$), the summation of $C(w)$ in Eqn. (1) reduces to a point:

$$p'_{k-2} = p_{k-3}N_{k-3}^3(w) + p_{k-2}N_{k-2}^3(w) + p_{k-1}N_{k-1}^3(w) + p_k N_k^3(w) \tag{2}$$

which is obviously interpolated by the curve $C(t)$ (see Fig. 1(a)). Conversely, if the point p_{k-2} is replaced by the point:

$$\frac{1}{N^3_{k-2}(w)}(-p_{k-3}N^3_{k-3}(w)+p_{k-2}-p_{k-1}N^3_{k-1}(w)-p_kN^3_k(w)) \tag{3}$$

in the sequence $[p_0, p_1, ..., p_m]$, the resulting curve would interpolate p_{k-2} itself (see Fig. 1(b)).

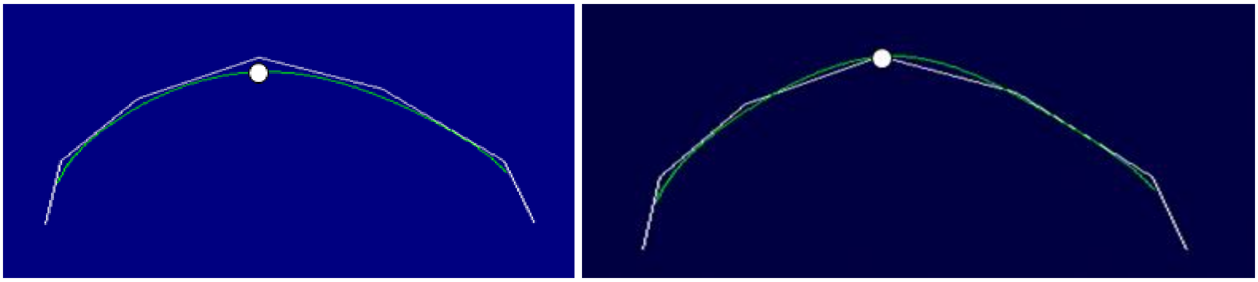

Fig. 1: Point Interpolation: (a) Point interpolated by the curve corresponding to parameter w, (b) Curve altered to interpolate the control point corresponding to parameter w.

B-Spline Surfaces
Given the usual knot vectors and the usual control point grid, a B-spline surface is defined by:

$$S(u,v) \equiv \sum_{i=0}^{m_1}\sum_{j=0}^{m_2} N_i^{p_1}(u)N_j^{p_2}(v)p_{ij} \tag{4}$$

Interpolating Iso-Parametric Curves
Assume that the given curve $C(t)$ is a cubic B-spline curve over a knot vector **T**, and also assuming that the given surface $S(u,v)$ is a bi-cubic B-spline surface over the knot vectors **U**×**V**, interpolating the curve by the surface is straightforward when the curve is iso-parametric with respect to the surface. For example, when the *uv* curve is a constant *v*-curve, i.e. $u(t) = t$ and $v(t) = c$, the surface curve $S(u(t), v(t))$ has the same *u*-basis as that of the surface; i.e. **U** = **T**.

In this case, for any parameter *u*, $N_i^{p_1}(u)$ may be factored out of the inner summation of (4), since it is constant along the *v* direction. Consequently, the global expression of (4) becomes:

$$\sum_{i=0}^{m_1} N_i^{p_1}(u)\sum_{j=0}^{m_2} N_j^{p_2}(v)p_{ij} \tag{5}$$

By comparison with Eqn. (4), Eqn. (5) represents a curve:

$$C'(u) \equiv \sum_{i=0}^{m_1} N_i^{p_1}(u)p'_i \tag{6}$$

where

$$p'_i = \sum_{j=0}^{m_2} N_j^{p_2}(v)p_{ij} \tag{7}$$

Moreover, the curve $C'(u)$ is obviously interpolated by the surface $S(u,v)$, something that is, again, made possible by the fact that the curve $C'(u)$ is iso-parametric with respect to parameter *u* of the surface.

In general, the summation of (7) will then depend on only p_2+1 terms;

$$p'_i = \sum_{j=k-p_2}^{k} N_j^{p_2}(v) p_{ij} \tag{8}$$

In the cubic case (where $p_2 = 3$), for example, the same reasoning of the curve case may be applied to deduce that the summation of (4) depends on at most 4 rows of the B-spline control point grid (see Fig. 2(a)).

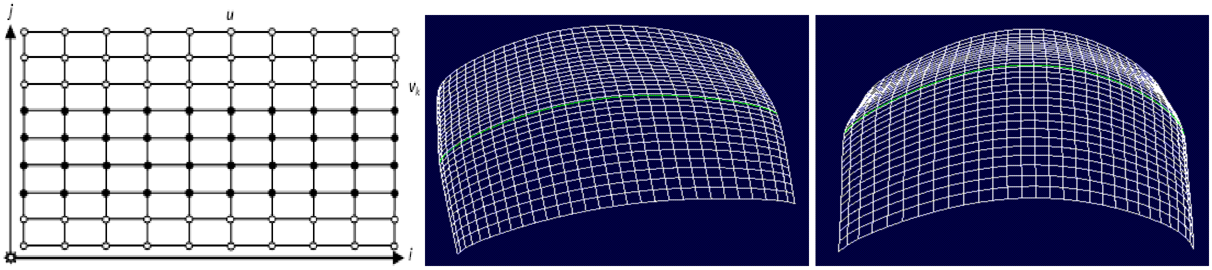

Fig. 2: Curve Interpolation: (a) B-Spline Polygonal Complex Marked on a B-spline Surface Grid, (b) Curve interpolated by the surface corresponding to parameter v_k, (c) Surface altered to interpolate the curve of the corresponding row of the polygonal complex.

The marked rows of Fig. 2(a) form what is called a B-spline *polygonal complex* [1, 11], which may also be expressed by a $4 \times m_1$ matrix M of points. Accordingly, the control polygon (P) represented by the following matrix multiplication:

$$\left[N_k^3(v) \quad N_{k\text{-}1}^3(v) \quad N_{k\text{-}2}^3(v) \quad N_{k\text{-}3}^3(v) \right] \times M \tag{9}$$

corresponds to a B-spline curve interpolated by the surface $S(u,v)$. Conversely, if the row of the grid corresponding to row number $k\text{-}2$ of M is replaced by the following polygon:

$$\frac{1}{N_{k\text{-}2}^3(v)} \left[\text{-}N_k^3(v) \quad \text{-}N_{k\text{-}1}^3(v) \quad 1 \quad \text{-}N_{k\text{-}3}^3(v) \right] \times M \tag{10}$$

then the resulting surface will interpolate the curve corresponding to this row.

Interpolating Non-Iso-Parametric Curves

In the non-iso-parametric case, the uv curve is constant neither along the u direction nor along the v direction. In other words, the knot vector **T** of the curve is neither equal to the knot vector **U** nor the knot vector **V** of the surface. This can come under a variety of forms; a simple one is depicted in Fig. 3(a).

The idea here is that no parameter of the surface would be constant along the vector **T** to make factorization of the summation of Eqn. (4) possible here. Consequently, one has to seek a wholly different formulation of the problem altogether to achieve the interpolation of the non-iso-parametric curve.

Mathematically, the problem may be re-formulated as follows: let $C(t)$ be our target B-spline curve whose pre-image in the uv parameter domain is a B-spline curve $\Phi(t) = \langle u(t), v(t) \rangle$. This is assumed to be non-iso-parametric with respect to a given tensor-product B-spline surface (see Fig. 3(a)).

The goal is to modify the surface $S(u,v)$ locally, so as to establish the following identity:

$$S(u(t), v(t)) = C(t) \tag{11}$$

where the curve on surface $S(u(t), v(t))$ would be equal to the following expression:

$$\sum\sum N_i(u(t)) N_j(v(t)) p_{ij} \tag{12}$$

Our interest in curves on surfaces here is just to provide a point of focus as to which parts of the surface would need to be modified in order to establish the identity describe by Eqn. (11).

As we mentioned above, the modification of the surface will be local. This is due to the locality of the B-spline basis functions. In fact, we only need to modify those control points for which the support of their corresponding B-spline basis functions intersects with the *uv* curve Φ(t). For example, if the surface is cubic by cubic and all interior knots are simple knots (multiplicity equal to one), then all interior bases $N_i^3(u)N_j^3(v)$ has the support of a 4×4 rectangle (see Fig. 3(b)) for this base. We shall mark the lower-left corner as corresponding to control point p_{ij} for this base.

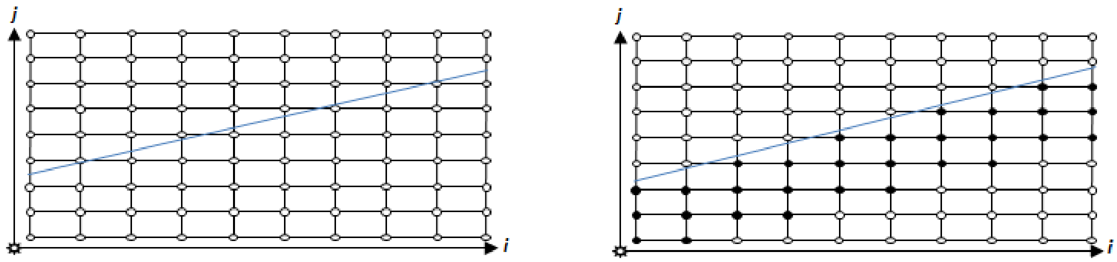

Fig. 3: Curve and Surface: (a) A Non-Iso-Parametric Curve on a B-spline Surface Grid, (b) Control Point Support of the Surface for the Curve.

For the given *uv* curve Φ(t), Fig. 3(b) shows the control points which are involved in the construction of the curve on surface *S(u(t), v(t))*. On this basis, it would seem reasonable to assume that these are the only control points of the surface that need to be repositioned in order to achieve interpolation.

In this sense, the area of the surface specified by these control points may be considered as a generalization of the B-spline polygonal complex as depicted in Fig. 2(a), for example. However, the absence of a constant parameter here means that factorization is not possible here.

Thus, in order to achieve interpolation, we need to make one step back. The obvious option here is to think of these control points as variables then use them is a system of linear equations, whose solution would satisfy the constraints specified by Eqn. (11); i.e., altering the surface to a situation that would interpolate the given non-iso-parametric curve.

The Proposed Solution

Starting from our goal which is to alter the surface so as to establish the identity expressed by Eqn. (11), the first step would be to make sure that the two curves *C(t)* and *S(u(t), v(t))* have the same degree.

In fact, if the initial curve *C(t)* is cubic and the initial surface *S(u, v)* is bi-cubic, and if the curves *u(t)* and *v(t)* are both linear, then the curve-on-surface *S(u(t), v(t))* will be of degree 6.

As a result, the degree of the curve *C(t)* should be elevated from 3 to 6 (see [14]), which will give us the curve *C'(t)* of degree 6 with knot vector **T'** and control point sequence $(p_k^{'})_k$.

$$C'(t) = \sum_k p_k^{'} N_k(t) \tag{13}$$

Next, with reference to Eqn. (12), and for each pair of indices <*i*, *j*>, there will be a sequence of coefficients $(\delta_{ijk})_k$ along the knot vector **T'** such that:

$$N_i(u((t))N_j(v(t)) = \sum_k \delta_{ijk} N_k(t) \tag{14}$$

This is a problem of B-spline multiplication and composition. More details about how that solution followed here may be found in Hu&Sun [6]. The full analysis is in E. T. Y. Lee [7] who gives a simple and quick blossoming-based algorithm to compute the B-spline coefficients from the power polynomial form of B-spline. More literature on the subject may be sought in Lyche and Morken [8], Morken [10] and Ramshaw [15].

The System of Linear equations

Now, we can rewrite the curve-on-surface expression in Eqn. (11) and Eqn. (12) as:

$$S(u(t), v(t)) \equiv \sum_i \sum_j N_i(u(t))N_j(v(t))\, p_{ij} \tag{15}$$

which is again: $S(u(t),v(t)) \equiv \sum_i \sum_j \sum_k \delta_{ijk} N_k(t) p_{ij}$ and again:

$$S(u(t),v(t)) \equiv \sum_k \sum_i \sum_j \delta_{ijk} N_k(t) p_{ij}$$

Now, if we match that against the degree-elevated B-spline curve *C'(t)*, we obtain:

$$\sum_i \sum_j \delta_{ijk} p_{ij} = p_k' \tag{16}$$

for all control points of the curve *C'(t)*.

Solving this linear system we can obtain the new positions of the surface control points (p_{ij}) in the support of the curve depicted in Fig. 3(b), which will guarantee that the curve will be interpolated by the modified version of the surface.

However, we should perhaps note that the system of equations (16) is not always solvable. In fact, the general condition for the solvability of Eqn. (16) is very complicated. For example, we cannot move a lower degree surface to a higher degree target curve in general or, in other words, we cannot move a surface with lower complexity to a target curve with higher complexity. For that, we may need to use Degree Elevation and Knots Insertion to increase the complexity of the surface.

For this reason, we seek the use of algorithms such as the SVD (Singular Value Decomposition, cf. Numerical Recipes in C) to solve Eqn. (16), see [3] and [5]. This algorithm can find the exact solution of the problem if there is one and the least square approximation if there is no exact one. Moreover, if there happens to be more than one solution, this algorithm can find one with minimum change from original control points.

Conclusions and Further Work:

This paper proposes a blossoming-based solution for the interpolation of non-iso-parametric B-spline curve by a B-spline surface. This solution is a slight adaptation of the solution proposed by Hu&Sun [6] used for trimmed surface matching which in turn used an algorithm by E.T.Y Lee [7] for handling the composition and product of B-splines.

This solution is of a general nature. For example, it can handle the interpolation of iso-parametric curves as simple particular case. Moreover, further to what is proposed in this paper, one can use it to handle the situation where the knot lines *u(t)* and *v(t)* of the curve could have degrees that are more than just one.

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