Title:
Fairness Metric of Plane Curves Defined with Similarity Geometry Invariants

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Introduction:
A curve is considered fair if it consists of continuous and few monotonic curvature segments. Polynomial curves such as Bézier and B-spline curves have complex curvature function hence the curvature profile may oscillate easily with a little tweak of control points. Thus, bending energy and shear deformation energy are common fairness metrics used to produce curves with monotonic curvature profiles. The fairness metrics are used not just to evaluate the quality of curves, but it also aids in reaching to the final design.

Curve synthesis is a process of generating curves with a well-defined Cesáro equation, which describes the curvature κ of a curve as a function of its arc length s. Log-aesthetic curves (LAC in short) [3] are generated with a Cesáro equation derived by letting the Logarithmic curvature graph (LCG) as a linear function with the gradient as a. This curve has gained its momentum in design environment and now is it used for automobile [4] and architecture [6] design. The family of LACs includes logarithmic (equiangular) curves ($\alpha = 1$), clothoid curves ($\alpha = -1$), circle involutes ($\alpha = 2$) and Nielsen's spiral ($\alpha = 0$). It is possible to generate and deform LACs in real time regardless of its integral forms using their unit tangent vectors as integrands when $\alpha \neq 1,2$.

Recently, Sato and Shimizu [5] expressed LACs by a simple equation in similarity geometry where the direction angle θ of a given curve is invariant. For a given curve $C(\theta) = (x(\theta), y(\theta))$, the similarity curvature $S(\theta) \equiv -\rho_\theta/\rho$ is also invariant where ρ is radius of curvature and $\rho_\theta = d\rho/d\theta$. Thus, the slope of the LCG of a LAC can be expressed by

$$\alpha = \frac{S_\theta}{S^2} + 1 \tag{1}$$

The similarity curvature of LAC satisfies the following Riccati (Bernoulli) equation:

$$S_\theta = (\alpha - 1)S^2 \tag{2}$$

The above equation can be solved easily to obtain the similarity curvature of LAC as follows:

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$$S(\theta) = \frac{-1}{(\alpha-1)\theta+c} \tag{3}$$

where c is an integral constant.

In this paper, we propose two types of fairness metric functionals to fair plane curves defined by the similarity geometry invariants, i.e. similarity curvature and its reciprocal to extend a variety of aesthetic fairing metrics. Section 4 also illustrates numerical examples to show how LACs changes depending on α and G^1 constraint. In section 5, we introduce an extra term for our functionals and symmetry concept for LAC.

Similarity Geometry:

We may deduce to figures similar each other when these figures possess the same shape even if their sizes are different. In similarity geometry if two objects are similar, then we deduce that both are equivalent. In Euclidean geometry, circles with different radii are considered different entity, but in similarity geometry circles with different radii are regarded as the same.

In this section, we derive similarity Frenet frame to introduce the definition of similarity curvature and show its role in similarity geometry [2]. Since we know that the arc length s may vary, thus the representation of plane curves is in the form of direction angle θ parameterized which is invariant by scaling. First, let a plane curve is given as a function of its arc length by

$$C(s) = (x(s), y(s)) \tag{4}$$

and its Frenet frame $F(s) = (T(s), N(s))$. We assume the curve is not a straight line and the direction angle θ is defined by

$$\theta = \int_0^s \kappa(s)\, ds. \tag{5}$$

Next, let tangent vector $T^{Sim}(\theta)$ as follows to define the Frenet frame in similarity geometry,

$$T^{Sim}(\theta) \equiv \frac{dC}{d\theta}(\theta). \tag{6}$$

Thus, we may simplify as

$$T^{Sim}(\theta) = \frac{dC}{ds}\frac{ds}{d\theta} = \frac{1}{\kappa(s)}T(s) \tag{7}$$

where $T(s)$ is the first derivative of $C(s)$ with respect to s and it is a unit tangent vector of the curve. Let $N^{Sim}(\theta)$ be

$$N^{Sim}(\theta) = \frac{1}{\kappa(s)}N(s) \tag{8}$$

Since $\det(T^{Sim}, N^{Sim}) = 1/\kappa^2$, hence $F^{Sim}(\theta) = (T^{Sim}(\theta), N^{Sim}(\theta))$ has a value in

$$CO^+(2) = \{X \in CO(2) |\, \det X > 0\} \tag{9}$$

where $CO^+(2)$ is a set of 2×2 real matrix A such that $AA^T = cE$ for an arbitrary constant c. Here A^T denotes a transpose of matrix A and E does a unit matrix. The derivatives of $T^{Sim}(\theta)$ and $N^{Sim}(\theta)$ are given by

$$\frac{d}{d\theta}T^{Sim}(\theta) = -\frac{\kappa_s(s)}{\kappa(s)^2}T^{Sim}(\theta) + N^{Sim}(\theta) \tag{10}$$

$$\frac{d}{d\theta}N^{Sim}(\theta) = -\frac{\kappa_s(s)}{\kappa(s)^2}N^{Sim}(\theta) - T^{Sim}(\theta) \tag{11}$$

From equation (10) and (11), we define

$$S(\theta) = \frac{\kappa_s(s)}{\kappa(s)^2} \tag{12}$$

Equation (12) is an invariant in similarity geometry and it is denoted as similarity curvature. Therefore, $F^{Sim}(\theta)$ satisfies the following differential equation:

$$\frac{d}{d\theta}F^{Sim}(\theta) = F^{Sim}(\theta)\begin{pmatrix} -S(\theta) & -1 \\ 1 & -S(\theta) \end{pmatrix} \tag{13}$$

.

The above equation is called the formula of Frenet frame in similarity geometry.

Similarity Geometry Invariants:

As stated in the previous section, by regarding direction angle θ as a function of arc length s, the similarity curvature $S(\theta(s))$ is defined by

$$S\big(\theta(s)\big) = \frac{1}{\kappa(s)^2}\frac{d\kappa}{ds} = -\frac{d\rho}{ds} \tag{14}$$

where κ is curvature. Similarity radius of curvature $V(\theta(s))$ is defined as a reciprocal of similarity curvature $S(\theta(s))$ and it is derived as follows

$$V\big(\theta(s)\big) = \frac{1}{S(\theta(s))} = \frac{\kappa(s)^2}{\frac{d\kappa}{ds}} = -\frac{1}{\frac{d\rho}{ds}} \tag{15}$$

In this paper, two types of functionals are proposed to fair a plane curve $C(t)$ whose domain is $[a, b]$. The first type is given by

$$F_{sc}\big(C(t)\big) = \int_a^b S(\theta(t))^2 \frac{d\theta}{dt}dt \tag{16}$$

and the second type is

$$F_{sroc}\big(C(t)\big) = \int_a^b V(\theta(t))^2 \frac{d\theta}{dt}dt \tag{17}$$

Eqns. (16) and (17) are now rewritten as follows:

$$F_{sc}\big(C(t)\big) = \int_0^l \frac{1}{\kappa(s)^4}\left(\frac{d\kappa}{ds}\right)^2 \kappa(s)ds = \int_0^l \frac{1}{\kappa(s)^3}\left(\frac{d\kappa}{ds}\right)^2 ds = \int_0^l \frac{1}{\rho(s)}\left(\frac{d\rho}{ds}\right)^2 ds \tag{18}$$

where l is a total length of curve $C(t)$. Similarly, the second type is rewritten as follows:

$$F_{sroc}\big(C(t)\big) = \int_0^l \frac{\kappa(s)^5}{\left(\frac{d\kappa}{ds}\right)^2}ds = \int_0^l \frac{1}{\rho(s)\left(\frac{d\rho}{ds}\right)^2}ds \tag{19}$$

Consider two traditional functionals commonly used for fairing plane curves:

$$\int_0^l \kappa^2(s)ds \tag{20}$$

called bending energy and

$$\int_0^l (\frac{d\kappa}{ds})^2 ds \tag{21}$$

called shear deformation energy. These functionals are clearly different from F_{sc} or F_{sroc}.

Euler-Lagrange Equations [1]:

Similarity Curvature
From equation (18):

$$F_{sc}\big(C(t)\big) = \int_0^l f_{sc}(s)ds = \int_0^l \frac{1}{\kappa(s)^3}\left(\frac{d\kappa}{ds}\right)^2 ds \tag{22}$$

Its Euler-Lagrange equation in terms of κ is

$$\frac{\partial f_{sc}}{\partial \kappa} - \frac{d}{ds}\frac{\partial f_{sc}}{\partial \dot{\kappa}} = -3\frac{\dot{\kappa}^2}{\kappa^4} - 2\frac{d}{ds}\frac{\dot{\kappa}}{\kappa^3} = \frac{3}{\kappa^4}\left(\dot{\kappa}^2 - \frac{2}{3}\kappa\ddot{\kappa}\right) = 0 \tag{23}$$

where $\dot{g} = dg/ds$ and $\ddot{g} = d^2g/ds^2$ for function g of s. constant.

It is known that LACs satisfy the following equation [3]:

$$\kappa^{-\alpha} = cs + d \tag{24}$$

where c and d are constants. We obtain Eqn. (25) after differentiating both sides of the above equations twice:

$$-\alpha(-\alpha - 1)\kappa^{-\alpha-2}\dot{\kappa}^2 - \alpha\kappa^{-\alpha-1}\ddot{\kappa} = 0 \tag{25}$$

If $\alpha \neq 0$ and $-\alpha - 1 \neq 0$ ($\alpha \neq -1$), then

$$\dot{\kappa}^2 - \frac{1}{\alpha+1}\kappa\ddot{\kappa} = 0 \tag{26}$$

By comparing Eqns. (23) and (26), the curve which minimizes Eqn. (22) is a log-aesthetic curve whose α is equal to 1/2. This fact demonstrates that LAC can be expressed by a simple similarity curvature, which has a natural property and plays an important role in similarity geometry.

On the other hand, from

$$F_{sc}\big(C(t)\big) = \int_0^l \frac{1}{\rho(s)}\left(\frac{d\rho}{ds}\right)^2 ds \tag{27}$$

its Euler-Lagrange equation in terms of ρ is

$$\frac{\partial f_{sc}}{\partial \rho} - \frac{d}{ds}\frac{\partial f_{sc}}{\partial \dot{\rho}} = -\frac{\dot{\rho}^2}{\rho^2} - 2\frac{d}{ds}\frac{\dot{\rho}}{\rho} = \frac{1}{\rho^2}(\dot{\rho}^2 - 2\rho\ddot{\rho}) = 0 \tag{28}$$

Similarly, Eqns. (24) and (25) are rewritten with ρ

$$\rho^{\alpha} = cs + d \tag{29}$$

and

$$\alpha(\alpha-1)\rho^{\alpha-2}\dot{\rho}^2+\alpha\rho^{\alpha-1}\ddot{\rho}=0 \tag{30}$$

are satisfied by LACs. If $\alpha \neq 0$ and $\alpha-1 \neq 0$ ($\alpha \neq 1$), then

$$\dot{\rho}^2+\frac{1}{\alpha-1}\rho\ddot{\rho}=0 \tag{31}$$

By comparing Eqns. (28) and (31), the curve which minimizes Eqn. (27) is a log-aesthetic curve whose α is equal to 1/2. This result is consistent to that of the curvature formulation shown above.

Similarity Radius of Curvature
From equation (19)

$$F_{sroc}\big(C(t)\big)=\int_0^l f_{sroc}(s)ds=\int_0^l \frac{\kappa(s)^5}{\left(\frac{d\kappa}{ds}\right)^2}ds \tag{32}$$

its Euler-Lagrange equation in terms of κ is

$$\frac{\partial f_{sroc}}{\partial\kappa}-\frac{d}{ds}\frac{\partial f_{sroc}}{\partial\dot{\kappa}}=\frac{\kappa^4}{\dot{\kappa}^2}+2\frac{d}{ds}\frac{\kappa^5}{\dot{\kappa}^3}=15\frac{\kappa^4}{\dot{\kappa}^4}\left(\dot{\kappa}^2-\frac{2}{5}\kappa\ddot{\kappa}\right)=0 \tag{33}$$

By comparing Eqns. (33) and (26), the curve which minimizes Eqn. (32) is LACs whose α is equal to 3/2. From

$$F_{sroc}\big(C(t)\big)=\int_0^l f_{sroc}(s)ds=\int_0^l \frac{1}{\rho(s)\left(\frac{d\rho}{ds}\right)^2}ds \tag{34}$$

its Euler-Lagrange equation in terms of κ is

$$\frac{\partial f_{sroc}}{\partial\rho}-\frac{d}{ds}\frac{\partial f_{sroc}}{\partial\dot{\rho}\kappa}=-\frac{3}{\rho^2\dot{\rho}^4}(\dot{\rho}^2+2\rho\ddot{\rho})=0 \tag{35}$$

By comparing Eqns. (35) and (31), the curve which minimizes Eqn. (34) is LACs whose α is equal to 3/2. Again this result is consistent to that of the curvature formulation.

<u>Numerical Examples:</u>
Figure 1. shows the comparisons of LAC shapes whose α=1/2 and 3/2. It consists of three pairs of LACs and the curves in each pair are generated with the same G^1 constraints. When the G^1 constraints vary drastically from those for a circular arc, then the LAC shapes become distinctly different. Although the differences of their shapes are somehow restricted, switching α from 1/2 to 3/2 and vice versa provides a subtle deformation of the curve.

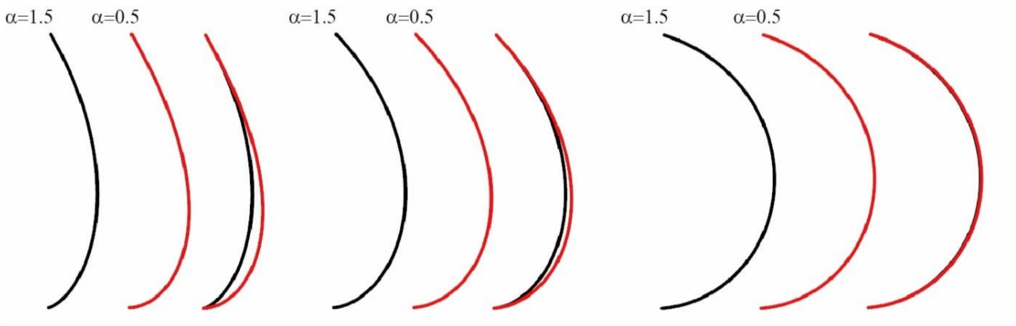

Fig. 1: Comparisons of LACs whose α=1/2 and 3/2.

Conclusions:
In this research, we have proposed two types of fairness metric functionals for fairing a plane curve defined by similarity curvature and similarity radius of curvature, which are invariant in similarity geometry. We have shown that by minimizing the integral of square of similarity curvature, we obtain LACs whose α equals to 1/2. Similarily for similarity radius of curvature, we obtain LACs for α equals to 3/2. Thus, a clear interpretation of the effect of the slope of the logarithmic curvature graph, especially when α is equal to 1/2 and 3/2 are derived.

We have extended our functionals to handle general LACs by introducing a power function of similarity curvature. The new functionals defined by similarity geometry invariants in Eqn. (4.15) is remarkably better than those previously proposed in [5, 10] because of its scale invariance. Furthermore, we have extended LAC by adding a term to the functional to suppress increase of the change of the direction angle of the curve and obtained quasi aesthetic curves and proposed σ curve to introduce symmetry concept for LAC. For future work, we would like to clarify the relationship between the quasi aesthetic and σ curves and to extend our fairing metrics for free-form surfaces.

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