Title:
Parametric Co-Design of Modular Free-Form 2-Manifolds

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Introduction:

For the Exhibition of Mathematical Art at the Joint Mathematics Meeting 2016 [2] I wanted to create a couple of sculptures that could serve as mathematical visualization models for non-orientable surfaces of higher genus, but would also hold up as aesthetically pleasing free-form sculptures in their own right. Inspired by project LEGO-Knots [4], [5], I planned to design a small set of modular components that could be assembled in many diverse ways to form single-sided (σ=1) as well as double-sided (σ=2) 2-manifolds of genus 2 and higher. The parts should be suitable for being built with Fused Deposition Modeling (FDM) or with Selective Laser Sintering (SLS), and they should result in sculptures that are about 1-2 feet tall.

The starting point for my design was the classical Klein-bottle. The key geometrical feature here is the Klein-bottle-mouth (KBM) at the top of Figure 1(a), where a thick tube turns outside-in, like a sock being inverted, and the thinner inner tube then emerges through the side-wall of the thicker tube. To allow the composition of surfaces of higher genus, this basic module has to be enhanced into a 3-way junction, where one of the three tubular ends exposes the opposite side of the surface from the one visible at the other two tubular ends (Fig.1b). This can be achieved in several different ways. Either the thick or the thin tube segments could be split into two branches outside the inverting KBM, or this split could occur in a more integrated way inside the KBM itself. I started out by sketching several possible geometries and contemplating which ones would lead to an attractive and relatively compact modular element. I soon rejected as too "cumbersome" any designs that looked like a combination of two KBMs in a single 3-way junction. I also eliminated the possibility of making a regular tubular 3-way junction and then inverting one or two of the tubular ends with a cross-cap-like pinch in the tube (Fig.1c), or with a split into 3-or more twisted ribbons (Fig.1d), as used in Roelofs' "Moebiustorus" [3].

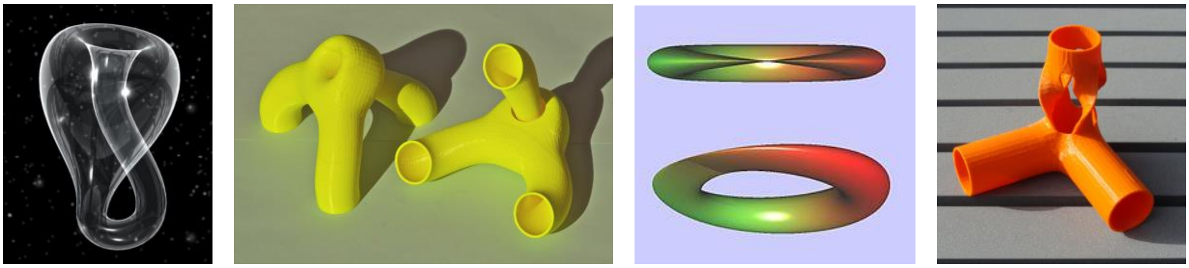
(a) (b) (c) (d)

Fig. 1: (a) Klein bottle; (b) 3-way KBM-junction; (c) surface-everting kink in pipe; (d) tube eversion via a split into individual twisted ribbons as in Roelofs' "Moebiustorus" [3].

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Four structures were selected to realize an integrated KBM-junction in different ways (Fig. 2). They all share a toroidal body as a common style element. In (a) the two "thick" branches merge into the toroidal ring, while a "thin" inverted tube emerges from the central tunnel of the torus. In (b) two thin tubes merge and jointly enter the torus tunnel. In (c) two thin tubes individually penetrate the ring of the torus, merge inside, and join up with the torus tunnel. In (d) the torus has been replaced with its convex hull, and the thick branch emerges along the central axis of this body, while two thin tubes enter separately through the perimeter, merge inside, and exit jointly in a KBM structure at the opposite side. All self-intersections, which are unavoidable when one tries to immerse a Klein-bottle in 3D Euclidean space, are eliminated by cutting suitable punctures into the surface; they are made just large enough to let a "thin" tube section pass through. Types (a) and (b) have only a single puncture, while (c) and (d) need two punctures.

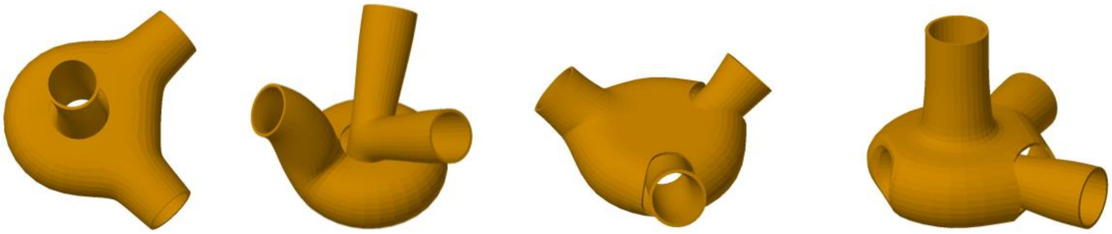

(a) (b) (c) (d)

Fig. 2: Four different ways of making a KBM-junction.

In order to make these modules fit together in many different ways, all tubular arms terminate with a consistent "standard arm-diameter." Moreover, the overall geometrical structure is a cubic graph (all nodes are 3-way junctions) based on some regular Platonic polyhedron (tetrahedron, cube, or dodecahedron), or based on a simple hosohedron structure with just two valence-3 junctions. The symmetry of these edge-graphs ensures that all the open tube pairs will join up with proper alignment. Eight modules, based on the set shown in Figure 2, will readily combine into the edge-frame of a cube, since the angles between any two tubular arms has been made exactly 90°.

The parts outlined in Figure 2 can be combined in hundreds of different ways. They have been designed so that the topological constraints of this project will be satisfied; but this does not guarantee that an aesthetically pleasing sculpture will result. For the latter goal, a complete assembly of eight KBM modules has to be evaluated. Only in this context can one decide what might be the optimal size of the polyhedral edge frame on which the sculpture is based, in relation to the exact shapes of the individual modules. In this global setting one would then like to adjust and fine-tune the relative distance between adjacent junction parts as well as the thickness of the connecting tubes between them, i.e., the "standard arm-diameter." All of this should be made possible without losing the modularity and general reconfigurability of the whole module set. Thus each of the module should adhere to a few global parameters that can be adjusted in this complete view of the sculpture. In addition, each module may have a few individual parameters that can be optimized once the global parameter values have been decided upon.

The design of the KBM modules to be fabricated thus turns out to be an iterative process. Some modules originally designed on an individual basis just did not want to fit into an overall satisfactory sculpture and had to be redesigned significantly or rejected. But the overall sculpture could not be visualized until a few of the individual modules had been designed in sufficient details and with the needed parameterization.

Even for a single object, creating a robust parametrization is a difficult task. As some parameters exceed their practically allowable range – rather than the range that is allowed on a given slider – some dependent values may become nonsensical, e.g., producing negative radii for cylinders or spheres. When a system is composed of mutually interacting components that are designed to be combined in many different ways, these problems become even more severe. This paper describes some of the geometrical details of this process, to serve as a guide for other situations where a modular set of parts has to be designed from scratch with the goal to allow a large number of different compositions.

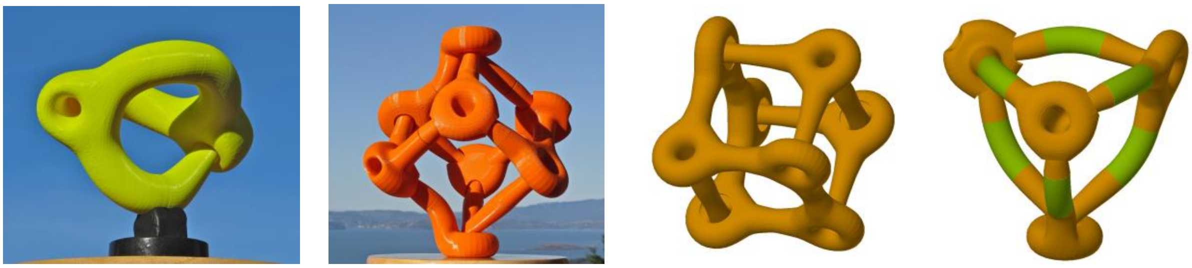
Modularization and Parameterization:

The geometrical modeling was structured to define for each KBM-junction module a coarse polyhedral mesh of the desired topology and of roughly the right geometry. This mesh is then refined by two or three steps of Catmull-Clark subdivision [1], and is subsequently turned into a material entity by forming an offset surface of appropriate thickness. Many CAD tools have difficulties performing subdivision and/or offsetting operations on single-sided, non-orientable starting meshes. Fortunately, all KBM components are well-behaved, orientable 2-manifolds with no self-intersections; the single-sidedness of the surface only emerges when the components are joined together.

The two parts shown in Figure 1(b) can be joined in three different ways; one of them results in a torus of genus 2, while the other two options form connected sums of two Klein bottles with two punctures each and a total genus of 4 (Fig.3a).

The parts shown in Figure 2 were designed to readily join together into a cube-frame structure. For most of the possible assemblies the result will be a single-sided ($\sigma=1$) surface of genus 10, corresponding to the connected sum of 5 Klein-bottles with a total of 12 punctures (Fig.3b). In a few instances when the "inside" tubes and "outside" tubes are carefully matched up, the result is a two-sided ($\sigma=2$), 5-hole torus of genus 5 (also with 12 punctures) (Fig. 3c).

To make a tetrahedral configuration, four new KBM modules could be built in which the angle between adjacent arms is only 60°. But a more modular approach is to introduce some additional curved connector parts that bend just the right amount (38.96°) to allow to re-use the 90° cube-corner modules in a tetrahedral configuration (Fig. 3d).



(a) (b) (c) (d)

Fig. 3: Assemblies of KBM-modules: (a) $\sigma=1$, $g=4$; (b) $\sigma=1$, $g=10$; (c) $\sigma=2$, $g=5$; (d) $\sigma=1$, $g=6$.

All KBM-junction modules have a toroidal body as a common style element, into which the three tubular arms can merge in different ways. One or two thicker tubes typically blend directly into the outer rim of this torus, while two thinner arms may merge either outside or inside the torus before they join up to the inside of the toroidal shell. Since the KBMs may want to be placed at different distances and angles from each other, but still need to be able to connect in a modular fashion, an overall regular polyhedral framework is first chosen for the whole sculpture, and the midpoints of all its edges are defined as the points where the tubular arms of adjacent KBM modules will join with a globally consistent arm-diameter. The tubular arms of the individual KBMs are realized as progressive sweeps. The endpoints of these tubular arms can readily be parameterized, so that on one end they will connect to the standardized joint location with the proper arm-diameter and tangent direction, and on the other end they will tie into the local geometry associated with the toroidal body of a particular KBM module type. (More details can be found in the full paper).

With the whole sculpture assembled in virtual form, all parameters can now be fine-tuned to give the overall most satisfactory result. Figure 4 illustrates the flexibility of the chosen parameterization in the context of a simpler tetrahedral framework. Some parameters control the overall network geometry: they define the overall size of the polyhedral edge frame (Fig.4b), the bulging of the connecting tubes between the different KBMs (Fig.4c), the positioning and tilting of the individual KBMs (Fig.4d), and the arm diameter at the tube junctions (Fig.4e).

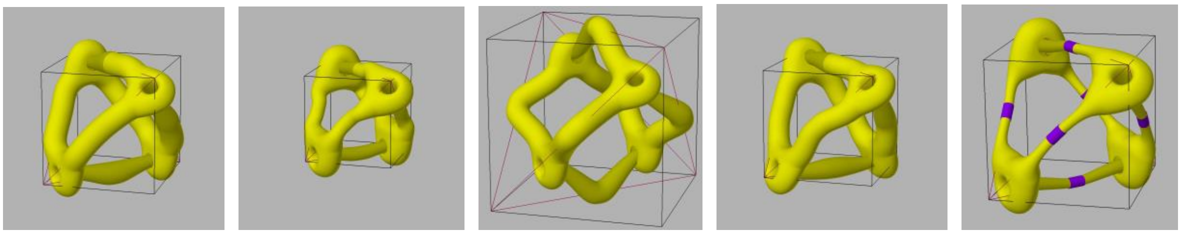

(a) (b) (c) (d) (e)

Fig 4: Change of framework: (a) default; (b) frame size; (e) arm bulge.; (d) KBM tilt; (e) arm diameter.

For each type of KBM module the parameterization further allows to adjust the two defining radii of the toroidal body (Fig.5b,c) to which the 3 tubular arms connect, as well as positioning and tilting of these torus shells (Fig.4b,d). In addition, the diameters with which the arms emerge from the junction geometry can be adjusted in response to these changes in the toroidal shell (Fig.5d). The end diameter of these arms is controlled by the global arm-parameter defined for the joints (Fig.4e,5e). (More details can be found in the full paper).

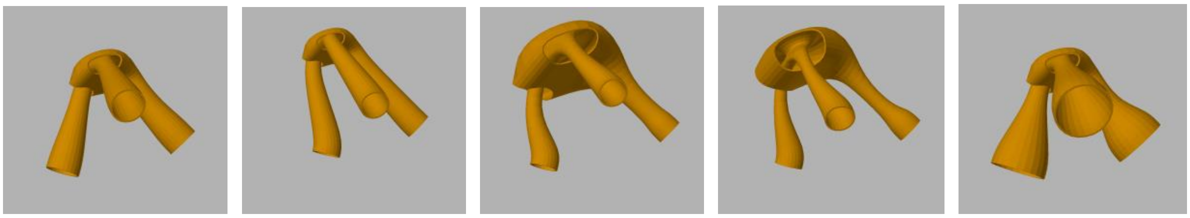
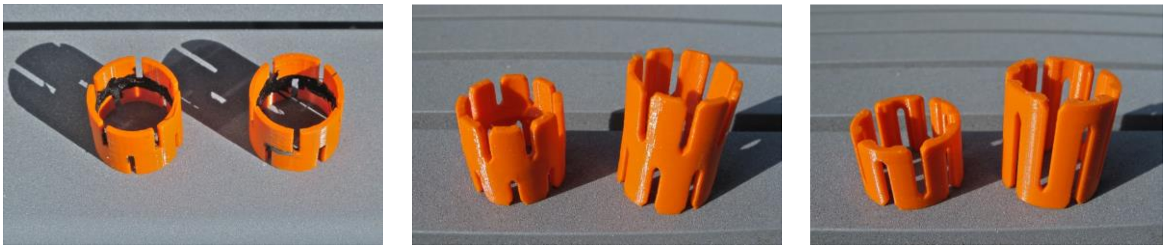

(a) (b) (c) (d) (e)

Fig. 5: Changing an individual KBM-module: (a) default; (b) KBM displacement; (c) adjusting size and tilt of toroidal body; (d) changing the inner arm-radius; (e) matching the arm radius at the joints.

Tube Connections:

The design of the structure shown in Figures 1b and 3a did not present much difficulty. This sculpture is composed of two identical components that end in three mutually parallel tube segments passing through the corners of an equilateral triangle. The lengths of the curved, connecting tube segments between the two KBM modules allowed for a very gradual transition from the thin tube emerging from the center of one toroidal ring into the larger tube branching into the outer surface of another torus. The two modules could readily be connected with three simple cylindrical connector pieces that were inserted by about half an inch into the open tube segments.

When I started the design of various KBM junction modules for a cube-based assembly (Fig. 3b,c), I did not yet have a clear understanding of what geometry would make the most robust connector element. So I designed the KBM modules with squarely cut off tubular arms, into which I could later insert appropriate tube connectors. I did not want to take the chance that, because of an inadequately designed connector geometry, I would later have to redo all the KBM modules with improved connectors. I just focused on designing the smoothest, most organic looking surface modules. Most of them involved some transitions from the “standard arm-diameter” at the joints between the modules to somewhat thinner tubes diving into the inner parts of the toroidal geometry that forms the basis for most of the Klein-bottle mouths, or towards some larger tube diameter leading to the outer envelope of those toroids. In some of the modules these transitions happen over a rather short distance – and this caused a problem: It made the tube segment rather conical right up to the joint, rather than nicely cylindrical. In some tubular arms the diameter increases as a distance from the joint; in others it decreases. This meant that a simple cylindrical connector piece can no longer make a good robust connection, and that a single connector type would not serve all possible combinations of the various conical tube segments. I ended up experimenting with quite a variety of tube connectors, giving them prongs that flare out by different amounts and have different stiffness (Fig. 6).



(a) (b) (c)

Fig. 6: Various connector types.

In hindsight, this approach turned out to be less than optimal. A better approach is to first design a robust connector system and then design each KBM module to fit smoothly into three of those connector geometries, properly placed at the mid-points of the edges of the overall chosen polyhedron geometry, as illustrated in Figure 4e. When designing a next set of modular parts, e.g., some 4-way junction modules that would fit an octahedral edge framework, I will definitely take this preferable approach. I will keep the arm diameters as constant as possible and then adjust the toroidal body, as well as any internal junction geometry, to accommodate the new tubular arms of more uniform thickness. (The impact on the overall aesthetic quality of the sculpture will have to be investigated!)

Summary and Conclusion:

Parametric design is powerful. It allows quick modification and fine-tuning of final results. As stated earlier, robust parametrization is difficult even for a single object, but becomes particularly hard, when a system is composed of several components that are designed to be combined in many different ways. This paper outlines this iterative design process for a small set of free-form modules that enable the composition of single-sided or double-sided 2-manifolds of higher genus. The two "Super-Bottles" shown in Figures 3a and 3b have been displayed in the Mathematical Art Exhibition of at the Joint Mathematics Meeting 2016 in Seattle [2].

The 3-way junction modules described above were optimized with an overall cube-frame structure in mind. If one now wants to extend the concept to a structure following a dodecahedral frame, the angles between the three tubular arms would be too small. New modules would have to be designed in which the arms spread by 108° between them. This may require more than just a parametric change; the polyhedral meshes defining the modules may have to be changed, e.g. a different set of faces on the toroidal shell would have to be removed to allow entry of one or two thin tubular arms at an optimal angle. In practice, appropriate parameterization for easy interactive optimization can only go so far; it can only cover a limited range of modifications. But within this range it offers a very powerful and convenient way to fine-tune and optimize a complex design.

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