Title:
Subdivision Based Piecewise C^2 Surfaces Construction for Meshes of Arbitrary Topology[1]

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Introduction:

It has been a long desire and a long effort of the computer graphics and geometric design community to have a nice approach to construct smooth surfaces from meshes of arbitrary topology. A nice approach should satisfy the following requirements:

- *simple*: no linear or non-linear system needs to be solved,
- *local*: changes to a control mesh only affect the resulting surface locally,
- *smooth*: the resulting surface is C2 everywhere, including at any extra-ordinary points,
- *convex*: the resulting surface satisfies the convex hull property,
- *explicit*: the resulting surface has an explicit expression of the form WMG for each patch, where W is a parameter vector, M is a constant matrix and G is the control point vector, so that surface evaluation, and computation of the first and second derivatives, normal and curvature at any point can be easily done from the simple representation.

When the degree (valence) of each vertex of the given mesh is 4, the algorithm for generating tensor product B-spline surfaces is such a nice approach. However, for meshes not in this category, as far as we know, there is no such an approach reported in the literature yet, although there are approaches that satisfy almost all of the above requirements [1,2,3,4,5,6,7,8,9,10]. In this paper we propose a new smooth surface construction technique that satisfies all the above requirements.

Previous Work:

Many researches have been performed to improve the smoothness of a CCSS at extraordinary points. Prautzsch [6] modifies the subdivision scheme near extraordinary points to generate a C^2 everywhere surface with zero curvature at extraordinary points. Zorin [9] and Levin [2] present schemes to yield a C^2 continuous surface by blending the limit surface with a low degree polynomial defined over the characteristic map in the vicinity of each extraordinary point. Loop and Schaefer [9] present a second order smooth filling of an n-valence Catmull-Clark spline ring with n biseptic patches, with shape optimization for free parameters. Peters and Karciauskas [5] introduce a guided subdivision scheme that uses a Bezier surface as a guide for each subdivision step, and a C^2 accelerated Bi-3 guided subdivision that uses 2^m subfaces in the m-th level for surface patches surrounding extraordinary points. In the second case, they show that although this scheme is not practical for Catmull-Clark sufaces, it can be applied to a polar configuration. However, these solutions are not completely satisfactory yet. Blending the limit surface with a precomputed curvature continuous surface patch is not flexible in surface representation. Filling the holes with bidegree-6 patches will result in higher

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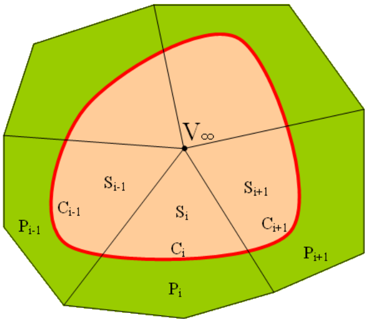
Gaussian curvature near the extraordinary points and make the limit surface unattactive. The bi-cubic subdivsion scheme that generates 2^m subpatches in the m-th subdivision is also undesired.

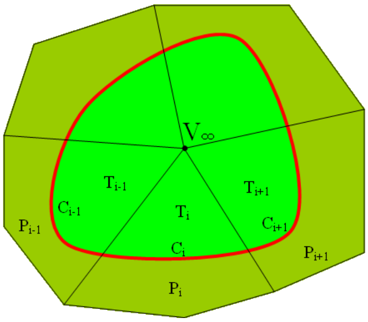
Basic Idea:
The basic idea of our approach is that for every patch P_i around an extra-ordinary vertex V of degree n, $1 \le i \le n$, we construct two C^2-continuous patches S_i and T_i (See Figure 1) in a way such that

- S_i is C^2-continuously connected with S_{i-1} and S_{i+1}, except at V_∞, where it is C^0,
- S_i is connected to P_i at C_i with C^2-continuity, where C_i is the intersection curve of S_i, T_i and P_i,
- T_i is C^2-continuously connected with T_{i-1} and T_{i+1},
- all T_i's are C^2-continuously connected at the extra-ordinary point V_∞,
- T_i is connected to P_i at C_i with C^0-continuity.

Note that if S_i and T_i are constructed this way, then a surface obtained by linearly blending S_i and T_i together is C^2-continuous everywhere. The key is how to construct S_i and T_i, for $1 \le i \le n$.



(a) Requirements for S_i



(b) Requirements for T_i

Fig. 1: Basic idea.

Construction of Si:
For a given mesh, we assume that all the faces are quadrilaterals and all the extra-ordinary vertices are separated by at least two faces. If it is not the case, simply perform (at most) two Catmull-Clark subdivisions to reach such a status. We consider all the patches P_i around an extra-ordinary vertex V of valance n, $1 \le i \le n$. It is well known that P_i depends on its surrounding $2n + 8$ vertices only [1].

Let $G_1 = [V,E_1,\cdots,E_n,F_1,\cdots,F_n,I_1,\cdots,I_7]^T$. Vertices for G_i can be identified similarly from the notation given in the paper [1]. Let

$$W(u,v) = [1,u,v,u^2,uv,v^2,u^3,u^2v,uv^2,v^3, u^3v,u^2v^2,uv^3,u^3v^2,u^2v^3,u^3v^3]. \tag{1}$$

Then P_i can be defined as follows.

$$P_i(u,v) = \begin{cases} \text{something we do not need}, & [0,\frac{1}{2}]\times[0,\frac{1}{2}] \\ W(2u-1,2v)M_4K_1AG_i, & [\frac{1}{2},1]\times[0,\frac{1}{2}] \\ W(2u-1,2v-1)M_4K_2AG_i, & [\frac{1}{2},1]\times[\frac{1}{2},1] \\ W(2u,2v-1)M_4K_3AG_i, & [0,\frac{1}{2}]\times[\frac{1}{2},1] \end{cases} \tag{2}$$

where M_4 is the B-spline tensor matrix of size 16×16, K_1,K_2,K_3 are constant picking matrices of size 16×24, each of which picks 16 proper vertices from the mesh if one subdivision is performed on patch P_i (See [1]). Matrix A is the extended Catmull-Clark subdivision matrix [1] which is of size $24 \times (2n + 8)$.

Now define $C_i(t) = P_i(\cos t,\sin t)$, $t \in [0,\pi/2]$. Let $L_i(r,t) = P_i(r\cos t,r\sin t)$. Then

$$L_i^r(1,t) = \frac{\partial L_i(r,t)}{\partial r}\Big|_{r=1},\ L_i^{rr}(1,t) = \frac{\partial^2 L_i(r,t)}{\partial r^2}\Big|_{r=1}$$

are the first and second derivatives of P_i at $C_i(t)$ with respect to r, respectively. Denote the limit point of V by V_∞. It is well known [1] that

$$V_\infty = \frac{1}{n(n+5)}(n^2 V + 4\sum E_i + \sum F_i)$$

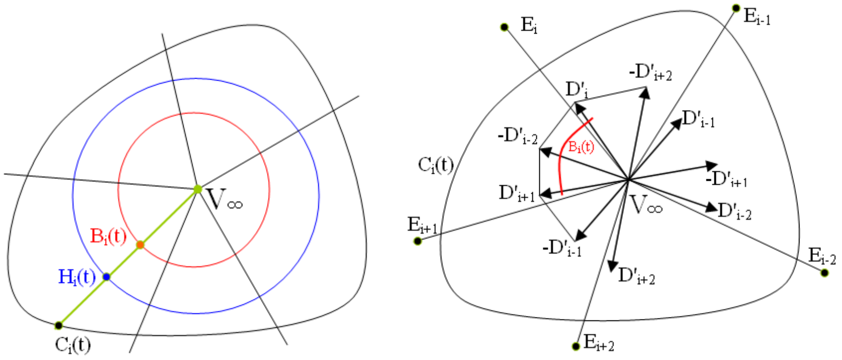
Let $R = [1,r,r^2,r^3]$, then we can construct a Bézier curve as follows such that it has the same first and second derivatives at $C_i(t)$ as those of P_i at $C_i(t)$.

$$S_i(r,t) = RM_b[V_\infty, L_i(1,t) - \tfrac{2}{3}L_i^r(1,t) + \tfrac{1}{6}L_i^{rr}(1,t), \\ L_i(1,t) - \tfrac{1}{3}L_i^r(1,t), L_i(1,t)]^T \tag{3}$$

where $0 \le r \le 1$, $0 \le t \le \pi/2$ and M_b is the Bézier matrix. If we plug L_i, L^r_i and L_i^{rr} into Eq. (3) and fully expand the formula, we get a matrix form representation for S_i as follows.

$$S_i(r,t) = \widetilde{W}(r,t)\widetilde{M}_n G_i, \; 0 \le r \le 1, 0 \le t \le \pi/2, \tag{4}$$

where An is a constant matrix of size $64 \times (2n + 8)$ and A_n can be pre-calculated for each n.



(a) Construction of T_i. (b) Construction of B_i when n is odd.

Fig. 2: Using Bézier curve to construct T_i.

Construction of Ti:

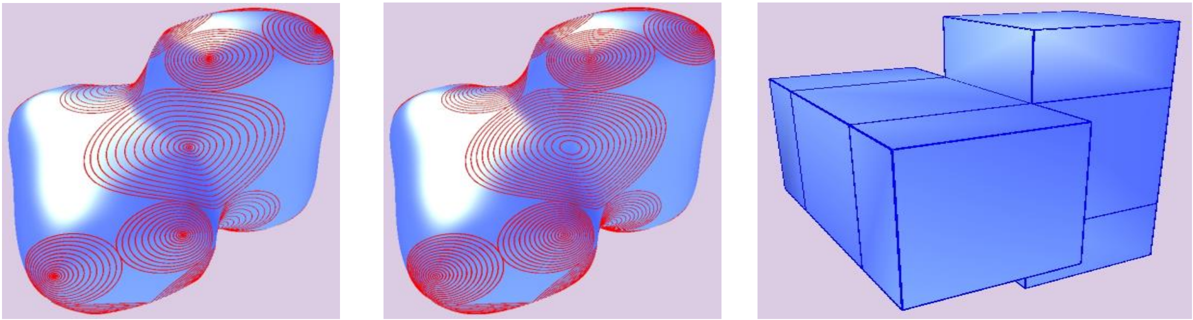
Recall that the requirements for the construction of T_i are that T_i itself has to be C^2 everywhere, C^2 with its neighboring patches T_{i-1} and T_{i+1} including at (0,0), and at least C^0 with $C_i(t)$. There are many ways to construct T_i. One simple way is to construct it as a Bézier patch, using an approach similar to the one given in the above section. For example, if we use two coplanar circles for all the $B_i(t)$'s and $H_i(t)$'s in the patches (see Figure 2) and let $R = [1,r,r^2,r^3]$, then the Bézier curve

$$T_i(r,t) = RM_b[V_\infty, B_i, H_i, C_i]^T, \quad 0 \le r \le 1,$$

becomes a surface when t varies, and this surface satisfies all the above requirements if the radius of H_i is two times the radius of B_i. Note that two Bézier curves constructed from $[V_\infty,B,H,C]$ and $[V_\infty, \hat{B}, \hat{H}, \hat{C}]$ are C^2 smoothly connected at V_∞ if and only if (1) B, V_∞, and $\hat{B}$ are collinear, (2) V_∞ is the midpoint of B and $\hat{B}$ and, (3) $\hat{H} = H + 4(V_\infty - B)$. The above defined $T_i(r, t)$ satisfies all the conditions because the two coplanar circles are smooth and symmetric with respect to V_∞. However, the resulting surface from this $T_i(r, t)$ may not be the one the designer wants. So we need more constraints on $B_i(t)$ and $H_i(t)$. In the following, we will construct a T_i that is similar to the original subdivision surface P_i at the extra-ordinary point by requiring that T_i and P_i have the same location, same first and second derivatives at V_∞.

The basic idea is again to construct Bézier curves that pass through V_∞ and have the same partial derivatives at V_∞ as P_i. This is done through four steps. First, we construct a B-spline curve $B_i(t)$ around the extra-ordinary point using the first partial derivative vectors along each edge of the extra-ordinary point. Second, we construct another B-spline curve $H_i(t)$ around the extra-ordinary point using the second partial derivative vectors along each edge of the extra-ordinary point. Third, find four control points for a Bézier curve such that it passes through V_∞ and $C_i(t)$, and such that its first derivative at V_∞ is $B_i(t)$ and the second derivative at V_∞ is $H_i(t)$. Finally, using the four points, we can construct a Bézier curve which becomes a smooth surface when t varies. Because $B_i(t)$, $H_i(t)$ and $C_i(t)$ are C^2 continuous, the constructed Bézier surface is C^2 smooth everywhere except at the extra-ordinary point. We can make it C^2 at the extra-ordinary point by adding one more condition such that $B_i(t)$ and $H_i(t)$ are symmetric with respect to the point V_∞.

(a) C^2 surface evaluation (b) CCSS evaluation (c) control mesh



(d) Isophotes on C^2 surface (e) Isophotes on CCSS surface (f) control mesh

Fig. 3: Test examples.

<u>Blending Ti with Si:</u>

To construct a C^2 patch $Q_i(r,t)$ in the ith face around an extra-ordinary vertex V of valance n, we first construct T_i and S_i using the methods given in the previous sections and then blend them together smoothly with a C^2 continuous blending function as follows.

$$\begin{aligned} Q_i(r,t) &= r^2 S_i(r,t) + (1-r^2)T_i(r,t) \\ &= r\widetilde{W}\widetilde{M}_n G_i + (1-r)\widehat{W}\widehat{M}_n G_i \\ &= \mathcal{W}\mathcal{M}_n G_i, \end{aligned} \tag{8}$$

where $0 \le r \le 1$, $0 \le t \le \pi/2$, W = [$W_t, rW_t, r^2W_t, r^3W_t, r^4W_t$] and M_n is a constant coefficient matrix of size $80 \times (2n+8)$. W_t is defined in section 4. M_n can be pre-computed for each n involved.

Now we can define a new C^2 patch $\widehat{P}_i(u,v)$ to replace the whole patch $P_i(u,v)$, as follows.

$$\widehat{P_i}(u,v) = \begin{cases} P_i(u,v), when\ u^2 + v^2 \geq 1, \\ Q_i(r,t), when\ u^2 + v^2 \leq 1, \end{cases} \tag{9}$$

where $0 \leq u,v \leq 1$ and $u = r\cos t, v = r\sin t$. It is clear that $\widehat{P_i}(u,v)$ is C^2 itself and C^2 with its neighboring patches, Note that from Eq. (2) one can see that $P_i(u,v)$, when $u^2 + v^2 >= 1$, can also be represented by a matrix form $W\overline{M_n}G_i$, where W is defined in section 4, $\overline{M_n}$ is a constant matrix of size $16 \times (2n+8)$ and can be pre-calculated as well. Hence at any parameter point (u,v), $\widehat{P_i}(u,v)$ and its derivatives can be calculated explicitly using just simple matrix operations.

Test Results:

The proposed approach has been implemented in C++ using OpenGL as the supporting graphics system on the Windows platform. Quite a few examples have been tested with the method described here (see Figure 3). All the examples have extra-ordinary vertices. With M_n pre-calculated for all different valences of n, the implementation is actually very easy. Although M_n is a big matrix, the computation needed for each point is not big at all because M_nG_i needs to be done only once. Our method is designed to ensure the resulting C^2 surface is similar to the subdivision surface. Figures 2(a-b) and 4(d-e) show two cases of comparison between a C^2 surface and its corresponding Catmull-Clark subdivision surface (CCSS). In either case, it is not obvious to tell the difference between the C^2 surface and its corresponding CCSS at all, although some very minor differences indeed exist. Figures 2(d-e) show the isophotes around extra-ordinary points using also our approach and CCSS approach. Ten isophotes are displayed around each extra-ordinary point and each isophote is corresponding to a circle in parameter space. The radii for the C^2 isophotes are the same as those for the CCSS isophotes. From these figures we can see that, when a point in the parameter space tends to (0, 0), the points generated by our approach are closer to the extra-ordinary point than points generated by a subdivision approach. When there are more points closer to the extra-ordinary point, there is more room for the generated surface to overcome the oscillation problem around an extra-ordinary point. As a result, our method produces smoother surface in the neighborhood of an extra-ordinary vertex.

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