Title:
Log-aesthetic Flow Governed by Heat Conduction Equations

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Introduction:
In this research, based on the concept of the smoothing by curve shortening flow and curvature flow, we will propose *log-aesthetic flow* to make free-form curves "log-aesthetic." We will discuss smoothing methods that deal with continuous curves as well as discrete ones.

Curve Shortening Flow and Curvature Flow [4]:
We deal with a curve $\boldsymbol{C}(p,t)$ defined by parameter p $(0 \leq p \leq 1)$ which deforms with time t. We assume that that its total length is a function of t and express it as $L(t)$. Then

$$L(t) = \int_0^1 \left\| \frac{\partial \boldsymbol{C}}{\partial p} \right\| dp \tag{1}$$

where $\|\boldsymbol{v}\|$ means the norm of vector $\boldsymbol{v}$. By differentiating the above equation with respect to t, we obtain

$$L'(t) = \int_0^1 \frac{\langle \frac{\partial \boldsymbol{C}}{\partial p}, \frac{\partial^2 \boldsymbol{C}}{\partial p \partial t} \rangle}{\left\| \frac{\partial \boldsymbol{C}}{\partial p} \right\|} dp \tag{2}$$

where $\langle \boldsymbol{a}, \boldsymbol{b} \rangle$ means the inner product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. By performing partial integration to Eqn.(2) ,

$$L'(t) = \left[\frac{\langle \frac{\partial \boldsymbol{C}}{\partial p}, \frac{\partial \boldsymbol{C}}{\partial t} \rangle}{\left\| \frac{\partial \boldsymbol{C}}{\partial p} \right\|} \right]_0^1 - \int_0^1 \langle \frac{\partial \boldsymbol{C}}{\partial t}, \frac{\partial}{\partial p} \left[\frac{\frac{\partial \boldsymbol{C}}{\partial p}}{\left\| \frac{\partial \boldsymbol{C}}{\partial p} \right\|} \right] \rangle dp \tag{3}$$

We assume that both of the end point positions of the curve are fixed with respect to time, i.e. $\partial \boldsymbol{C}(0,t)/\partial t = \partial \boldsymbol{C}(1,t)/\partial t = 0$. Then

$$L'^{(t)} = -\int_0^{L(t)} \langle \frac{\partial \boldsymbol{C}}{\partial t}, \kappa N \rangle ds \tag{4}$$

where s is an arc length and it is given by $ds = \|\partial C/\partial p\| dp$. κ and $\boldsymbol{N}$ are the curvature and the normal vector, respectively and they are defined by $\kappa \boldsymbol{N} = \partial^2 \boldsymbol{C}/\partial s^2$. Hence when

$$\frac{\partial \boldsymbol{C}}{\partial t} = \kappa N, \tag{5}$$

then $L(t)$ will decrease the most quickly. This flow is called *curve shortening flow*.

Here we define the curve's energy as $E(t)=\int_0^{L(t)}\kappa^2 ds$. Then

$$E'(t)=2\int_0^{L(t)}\kappa\frac{\partial\kappa}{\partial t}ds, \tag{6}$$

Therefore E(t) will decrease the most rapidly when $\partial\kappa/\partial t=-2\kappa$. This flow also deforms the shape of the curve using curvature and it is called *curvature flow*.

Log-aesthetic Flow

Arc-length Functional of the Log-aesthetic Curve in Aesthetic Space

The functional of the log-aesthetic curve which satisfies $\sigma=\rho^\alpha=cs+d$ is given by the following expression [5]:

$$J(t)=\int_0^L\sqrt{1+\sigma_s^2}ds. \tag{7}$$

Hence

$$J'(t)=\int_0^L(1+\sigma_s^2)^{-\frac{1}{2}}\sigma_s\sigma_{st}ds\ =\left[(1+\sigma_s^2)^{-\frac{1}{2}}\sigma_s\sigma_t\right]_0^L-\int_0^L\frac{\sigma_{ss}}{(1+\sigma_s^2)^{\frac{3}{2}}}\sigma_t ds \tag{8}$$

We assume that both of the curvatures at the end points are fixed with respect to time, i.e. $\partial\sigma(0,t)/\partial t=\partial\sigma(L,t)/\partial t=0$. Then

$$J'(t)=-\int_0^L\frac{\sigma_{ss}}{(1+\sigma_s^2)^{\frac{3}{2}}}\sigma_t ds \tag{9}$$

Then J(t) will decrease the most rapidly when

$$\sigma_t=\frac{\sigma_{ss}}{(1+\sigma_s^2)^{\frac{3}{2}}} \tag{10}$$

In this paper we call this type of the flow *length-based log-aesthetic flow.*

Energy Functional of the Log-aesthetic Curve in Aesthetic Space

It is known that the problem to minimize the length of a curve is equivalent to that to minimize its energy [2]. The energy of the log-aesthetic curve corresponding to Eqn. (7) [5] is given by

$$J_E(t)=\frac{1}{2}\int_0^L(1+\sigma_s^2)ds. \tag{11}$$

From the above equation and the assumption that $\partial\sigma(0,t)/\partial t=\partial\sigma(L,t)/\partial t=0$, we obtain

$$J'_E(t)=-\int_0^L\sigma_{ss}\sigma_t ds. \tag{12}$$

Therefore J_E(t) will decrease the most rapidly when

$$\sigma_t=\sigma_{ss}. \tag{13}$$

The above equation approximates Eqn. (10) and we call this type of the flow *energy-based log-aesthetic flow.*

Continuous Log-aesthetic Flow:

The heat conduction equation in one dimension is generally given by the following equations [3]:

$$\frac{\partial u}{\partial t}=a\frac{\partial^2 u}{\partial s^2},\qquad 0<s<L,\qquad 0<t \tag{14}$$

where u is an unknown function representing temperature and s is a parameter representing position. $a>0$ and a is called thermal conductivity. L is the total length of the object to be analized. Hence Eqn. (15) can be interpreted as a heat conduction equation with $a=1$ on $\sigma=\rho^\alpha=\kappa^{-\alpha}$ of a curve and we can describe the change of curvature, i.e. the deformation of the curve as the temperature change by heat conduction. This fact means that the energy-based log-aesthetic flow basically behaves as heat conduction and by solving the heat conduction equations we can obtain the shape of the curve deformed continuously by log-aesthetic flow. We will discuss how to solve Eqn. (14) under various conditions below.

In Case where $\boldsymbol{\kappa}=\mathbf{0}$ *at Both of the End Points*

Here we assume that $\alpha=-1$ and Eqn. (14) becomes

$$\kappa_t=\kappa_{ss}. \tag{15}$$

Furthermore we assume that the curvatures at both of the end points of the curve are equal to 0 and the total length of the curve is L. The initial and boundary conditions are given by $\kappa(s,0) = f(s)$ and $\kappa(0,t) = \kappa(L,t) = 0$, respectively. We assume that a solution $\kappa(s,t)$ is the product of $S(s)$ of only parameter s and $T(t)$ of only parameter t as

$$\kappa(s,t) = S(s)T(t). \tag{16}$$

From the above discussion and the principle of superposition, the general solution $\kappa(s,0)$ is given by

$$\kappa(s,t) = \sum_{m=1}^{\infty} a_m \sin\left(\frac{m\pi}{L}s\right) \exp(-\frac{m^2\pi^2}{L^2}t) \tag{17}$$

Therefore a_m is given by

$$a_m = 2\int_0^L \sin\left(\frac{m\pi}{L}s\right) f(s)ds, \qquad m = 1,2,\cdots \tag{18}$$

For example, when $L = 1$ and $f(s) = \sin(\pi s)$, from Eqn. (18)

$$a_m = \begin{cases} 1, & m = 0 \\ 0, & m > 1 \end{cases} \tag{19}$$

and

$$\kappa(s,t) = \sin(\pi s)\exp(-\pi^2 t) \tag{20}$$

Hence as t approaches infinity, $\kappa(s) \to 0$ and the curve converges to a straight line.

In Case where $\boldsymbol{\kappa} \neq \mathbf{0}$ *at the End Points: inhomogeneous boundary conditions*
Again we assume that $\alpha = -1$ and the curvatures of the end points of the curves are fixed κ_0 and κ_1, respectively. We define a function $\gamma(s,t)$ whose curvatures of the end points are equal to be 0 as follows:

$$\gamma(s,t) = \kappa(s,t) - \frac{1}{L}(\kappa_0(L-s) + \kappa_1 s) \tag{21}$$

Using Eqn. (17), the following general equation is obtained:

$$\gamma(s,t) = \sum_{m=1}^{\infty} a_m \sin\left(\frac{m\pi}{L}s\right) \exp(-\frac{m^2\pi^2}{L^2}t) \tag{22}$$

Therefore

$$\kappa(s,t) = \gamma(s,t) + \frac{1}{L}(\kappa_0(L-s) + \kappa_1 s) \quad = \sum_{m=1}^{\infty} a_m \sin\left(\frac{m\pi}{L}s\right) \exp(-\frac{m^2\pi^2}{L^2}t) + \frac{1}{L}(\kappa_0(L-s) + \kappa_1 s) \tag{23}$$

As t approaches infinity, $\kappa(s) \to (\kappa_0(L-s) + \kappa_1 s)/L$, the curvature of the curve is given by a linear function of s and the curve converges to a clothoid curve. For example, when $= 1$, $f(s) = \sin(\pi s)$, $\kappa_0 = 1$, and $\kappa_0 = 1$, from Eqn. (17),

$$a_m = \begin{cases} 1, & m = 0 \\ 0, & m > 1 \end{cases} \tag{24}$$

and

$$\kappa(s,t) = \sin(\pi s)\exp(-\pi^2 t) + 1 - s \tag{25}$$

Fig. 1. shows curvature distributions and the shapes of the curves deformed by log-aesthetic flow. Hence as t approaches infinity, $\kappa(s) \to 0$ and the curve converges to a straight line. The bottom of the right figure shows curves that are modified to pass through the given end points of the initial curve by iterative processes by shortening the curve to satisfy the given curvatures. Hence our method can generate a curve between given two points and two end curvatures. We will explain the details of the processes in the final paper.

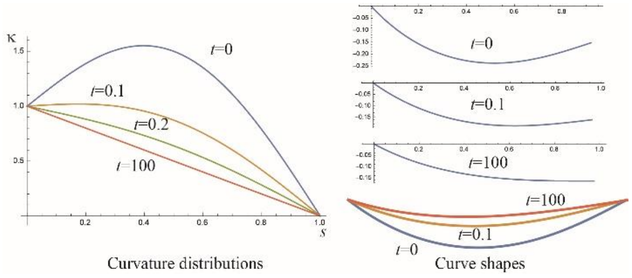


Fig. 1: Curvature distributions and their curve shapes deformed by log-aesthetic flow.

General $\boldsymbol{\alpha}$ *Case*

We deal with the following problem: Assume that the total length of the curve is L and the initial condition is $\kappa(s,0) = f(s)$. Furthermore, the curvatures of the end points of the curves are fixed κ_0 and κ_1, respectively. Then

$$\frac{\partial \sigma}{\partial t} = \frac{\partial^2 \sigma}{\partial s^2}, \qquad 0 < s < L, 0 < t, \qquad \sigma(s,0) = f(s)^{-\alpha},\ \ \sigma(0,t) = \kappa_0^{-\alpha}, \sigma(L,t) = \kappa_1^{-\alpha}. \qquad (26)$$

By solving the above partial differential equation, $\sigma(s,t)$ is obtained and $\kappa(s,t)$= $\sigma(s,t)^{-\frac{1}{\alpha}}$ is determined. For example, when $\alpha = -1/2$, $= 1$, $f(s) = \sin^2(\pi s)$, $\kappa_0 = 1$, and $\kappa_1 = 1$, from Eqn. (25),

$$\sigma(s,t) = \sin(\pi s)\exp(-\pi^2 t) + 1 - s \qquad (27)$$

So the curvature $\kappa(s,t)$ is given by

$$\kappa(s,t) = \{\sin(\pi s)\exp(-\pi^2 t) + 1 - s\}^2 \qquad (28)$$

Fig. 2 shows the curvature distributions of the deformed curves the shapes of the curves deformed by log-aesthetic flow.

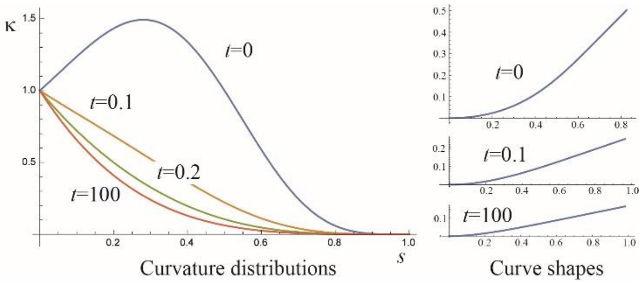
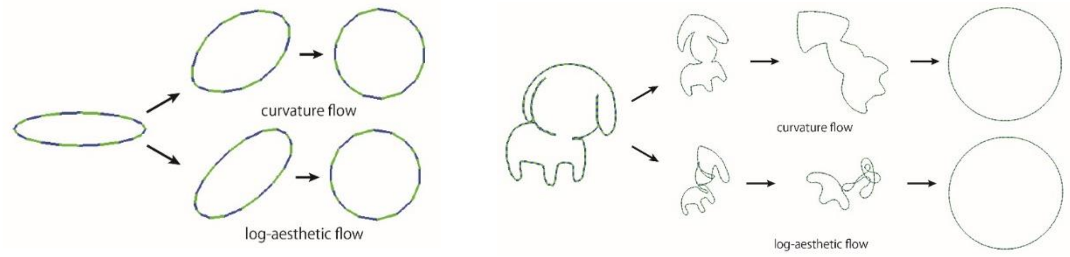


Fig. 2: Curvature distributions and their curve shapes deformed by log-aesthetic flow.

<u>Discrete Log-aesthetic Flow:</u>

In this section we discuss smoothing of discretely defined free-form curves, or polylines. Based on the log-aesthetic flow based on the energy, we update the positions of the vertices of a polyline by the discretized partial differential equation derived from Eqn. (14). Our algorithm uses the method proposed by Crane et al. [1] and we will fully explain the details of our algorithm in the final paper.

Fig. 3(a) compares deformations induced by log-aesthetic flow with those by curvature flow. Although the curve converges more quickly by curvature flow than by log-aesthetic flow, their final shapes are the same: a circle.



(a) An ellipse (b) A more complicated shape

Fig. 3: Smoothing shapes by curvature flow and log-aesthetic flow.

Fig. 3(b) shows a comparison between curvature flow and log-aesthetic flow applied to a more complicated. The final shapes are also circles in this case. If we interpret that the curve deformation is induced by log-aesthetic flow by heat conduction, the result is very natural because if we have a circular metal rod and keep one point of the rod at some temperature, the heat is conducted all over the rod and the temperature will be the same everywhere.

Conclusions and Future Work:

In this research, we have proposed the concept of log-aesthetic flow to make free-form curves "log-aesthetic" based on curve shortening flow and curvature flow. We have proposed new smoothing methods that can handle analytically defined continuous curves as well as discrete polylines. Smoothing by curvature flow is very popular among CG and CAD communities and log-aesthetic flow will be another choice for smoothing based on a physical law different from that of curvature flow. We have one degree of freedom α to control smoothing for log-aesthetic flow and can expect the completely smoothed shape of a given curve, which means the shape obtained by fairing, to be log-aesthetic curve. Since log-aesthetic flow is basically governed by the heat conduction equation, which has been well studied in both physics and mechanical engineering, its effects are easily inferred by the designers and we hope that it will be useful for practical aesthetic design. For future work, we would like to extend our method based on log-aesthetic flow for space curves and free-form surfaces.

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