Title:
Quadratic Log-Aesthetic Curves

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Introduction:

This paper proposes quadratic log-aesthetic curves, which are curves whose logarithmic curvature graphs [6] are quadratic. In previous work, generalized log-aesthetic curves [1,4] have been proposed by Miura and Gobithaasan. Generalized log-aesthetic curves are derived by shifting either the curvature or the radius of curvature of log-aesthetic curves [3,5]. Quadratic log-aesthetic curves are another generalization of log-aesthetic curves by making logarithmic curvature graphs quadratic.

We derive the equations of quadratic log-aesthetic curves and clarify their characteristics. One notable difference from generalized log-aesthetic curves is that quadratic log-aesthetic curves include curves with finite arc lengths and their curvature varying from 0 to infinity. We show that such curves can be obtained if the quadratic coefficient γ in the logarithmic curvature graph is negative. For drawing quadratic log-aesthetic curves, we need to compute the inverses of the error and imaginary error functions. We present a method for computing these inverses and confirm that the curves can be generated in real time.

Review of Error and Imaginary Error Functions:

This section briefly reviews the error and imaginary error functions [2], which will be necessary for drawing quadratic log-aesthetic curves. The error function $\operatorname{erf}(z)$ and imaginary error function $\operatorname{erfi}(z)$ are defined as

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} \mathrm{d}t, \quad \operatorname{erfi}(z) = -i \operatorname{erf}(i\, z) \tag{2.1}$$

where i is the imaginary unit. Figure 1 shows the error and imaginary error functions.

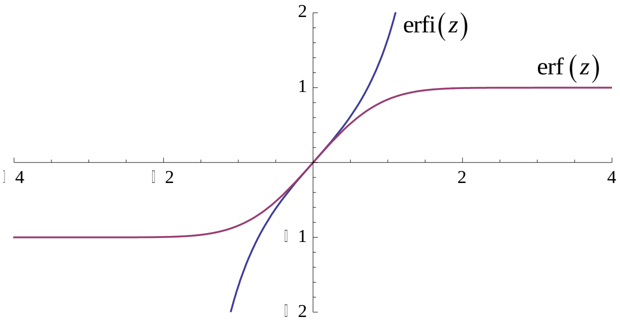

Fig. 1: The error and imaginary error functions.

As $|z|$ approaches 2, $\mathrm{erf}(z)$ approaches ± 1. In the double precision, if $|z|$ is greater than approximately 5.92, $\mathrm{erf}(z)$ is becomes essentially 1. Thus we safely assume $|z| \le 5$ when computing $\mathrm{erf}(z)$. When computing $\mathrm{erfi}(z)$, we have to be careful so that $|\mathrm{erfi}(z)|$ is within the range of the double precision. Therefore, we assume $|z| \le 26$ when computing $\mathrm{erfi}(z)$. These assumptions are used as the bounds for computing inverses of these functions, which are necessary for drawing quadratic log-aesthetic curves.

Quadratic Logarithmic Curvature Graphs and the Fundamental Equation:

Fig. 2(a) shows the linear logarithmic curvature graph with its slope $\alpha = 1$. Here, ρ is the radius of curvature and s is the arc length. Log-aesthetic curves are curves with linear logarithmic curvature graphs. See [3,5,6] for more details of log-aesthetic curves and logarithmic curvature graphs.

Quadratic log-aesthetic curves are curves whose logarithmic curvature graphs are quadratic as shown in Fig. 2 (b) or (c). The fundamental equation for quadratic log-aesthetic curves is

$$\log\left(\rho \frac{\mathrm{d}s}{\mathrm{d}\rho}\right) = \gamma(\log\rho)^2 + \alpha\log\rho + c \tag{3.1}$$

where γ, α are quadratic and linear coefficients respectively and c is a constant. Eqn. (3.1) is the fundamental equation for quadratic log-aesthetic curves. Note that Eqn. (3.1) represents only a parabola in the logarithmic curvature graph, including a line if $\gamma = 0$. Sincet other types of quadratic curves does not guarantee the monotonicity of the curvature, we need to consider parabolas only.

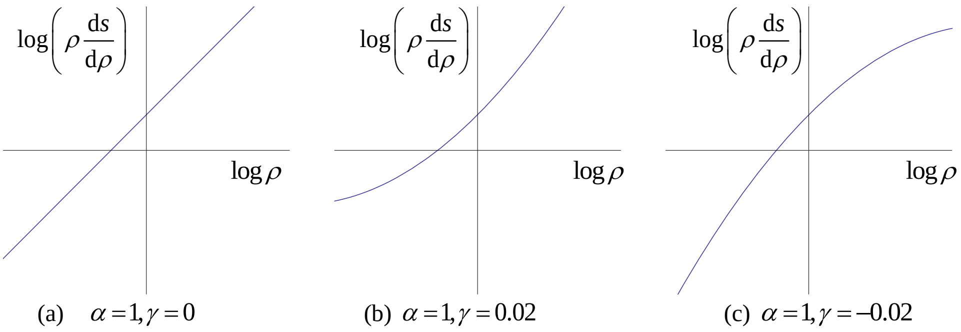

Fig. 2: Linear and quadratic logarithmic curvature graphs.

Deriving the Equations for Quadratic Log-Aesthetic Curves

This section derives the curvature κ of quadratic log-aesthetic curves as a function of arc length s to draw the curves using Frenet–Serret formulas. We rewirte Eqn. (3.1) in terms of curvature κ, we obtain

$$\log\left(-\kappa \frac{\mathrm{d}s}{\mathrm{d}\kappa}\right) = \gamma(\log\kappa)^2 - \alpha\log\kappa + d\ . \tag{4.1}$$

Eqn. (3.1) and Eqn. (4.1) are essentially the same equations. Taking the exponentials of both sides of Eqn. (4.1), we get

$$\frac{\mathrm{d}s}{\mathrm{d}\kappa} = -\frac{\kappa^{-\alpha+\gamma\log\kappa-1}}{\Lambda} \tag{4.2}$$

where $\Lambda = e^{-d}$. The reason Λ is set to like this is that the curve becomes a circular arc if $\Lambda = 0$ [5]. Integrating Eqn. (4.2) with respect to κ, we get the function $s(\kappa)$. Solving $s(\kappa)$ for κ and adding constants so that κ becomes 1 at $s = 0$, we get

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$$\kappa(s)=\begin{cases} e^{\frac{\alpha+2\sqrt{\gamma}\,\mathrm{erfi}^{-1}\left(-\frac{2\sqrt{\gamma}e^{\frac{\alpha^2}{4\gamma}}\Lambda s+\sqrt{\pi}\,\mathrm{erfi}\left(\frac{\alpha}{2\sqrt{\gamma}}\right)}{\sqrt{\pi}}\right)}{2\gamma}} & \text{if } \gamma>0 \\ e^{\frac{\alpha-2\sqrt{-\gamma}\,\mathrm{erf}^{-1}\left(\frac{-2\sqrt{-\gamma}e^{\frac{\alpha^2}{4\gamma}}\Lambda s+\sqrt{\pi}\,\mathrm{erf}\left(\frac{\alpha}{2\sqrt{-\gamma}}\right)}{\sqrt{\pi}}\right)}{2\gamma}} & \text{if } \gamma<0 \\ (1+\alpha\Lambda s)^{-\frac{1}{\alpha}} & \text{if } \alpha\neq 0 \text{ and } \gamma=0 \\ e^{-\Lambda s} & \text{if } \alpha=0 \text{ and } \gamma=0. \end{cases} \tag{4.3}$$

In Eqn. (4.3), the cases of $\gamma=0$ are exactly the same as the ones in log-aesthetic curves. If $\gamma>0$ or $\gamma<0$, inverse imaginary function $\mathrm{erfi}^{-1}(x)$ or inverse error function $\mathrm{erf}^{-1}(x)$ arises. For computing the inverse of these functions, we use a hybrid method combining the bisection method and the Newton's method. In the region where $|z|$ in Eqn. (2.1) is near 0 where the Newton's method is reliable, we use the Newton's method. Otherwise, the bisection method is used. Once Eqn. (4.3) is derived, the curve can be drawn by integrating the Frenet–Serret formulas with an initial condition. Similarly as in [5], we set the tangent and the point at $s=0$ as $[1 \quad 0]^{\mathrm{T}}$ and the origin, respectively.

Putting $\kappa=0$ and $\kappa=\infty$ into $s(\kappa)$ equation, we get

$$s(0)=\begin{cases} \infty & \text{if } \gamma>0 \\ \dfrac{e^{-\frac{\alpha^2}{4\gamma}}\sqrt{\pi}\left(1+\mathrm{erf}\left(\frac{\alpha}{2\sqrt{-\gamma}}\right)\right)}{2\sqrt{-\gamma}\Lambda} & \text{if } \gamma<0 \\ \infty & \text{if } \alpha\geq 0 \text{ and } \gamma=0 \\ -\dfrac{1}{\alpha\Lambda} & \text{if } \alpha<0 \text{ and } \gamma=0 \end{cases} \tag{4.4}$$

$$s(\infty)=\begin{cases} -\infty & \text{if } \gamma>0 \\ \dfrac{e^{-\frac{\alpha^2}{4\gamma}}\sqrt{\pi}\left(-1+\mathrm{erf}\left(\frac{\alpha}{2\sqrt{-\gamma}}\right)\right)}{2\sqrt{-\gamma}\Lambda} & \text{if } \gamma<0 \\ -\dfrac{1}{\alpha\Lambda} & \text{if } \alpha>0 \text{ and } \gamma=0 \\ -\infty & \text{if } \alpha\leq 0 \text{ and } \gamma=0. \end{cases} \tag{4.5}$$

$s(0)$ and $s(\infty)$ are the upper and lower bounds of arc length s depending on the value of α, γ and Λ. These bounds are necessary for drawing quadratic log-aesthetic curves. If $s(0)=\infty$, it means that the arc length to the point at $\kappa=0$ is infinity. Thus, the inflection point does not exist. In other words, the inflection point is at infinity. Similarly, if $s(\infty)=\infty$, the arc length to the point at $\kappa=\infty$ is infinity.

Results:

Fig. 3 shows quadratic log-aesthetic curves with various α,γ and their logarithmic curvature graphs and curvature plots. In all of Fig. 3, Λ is set to 1. In each of (a) to (f), the upper left figure is a quadratic log-aesthetic curve, the upper right is the logarithmic curvature graph, and the bottom is the curvature plot.

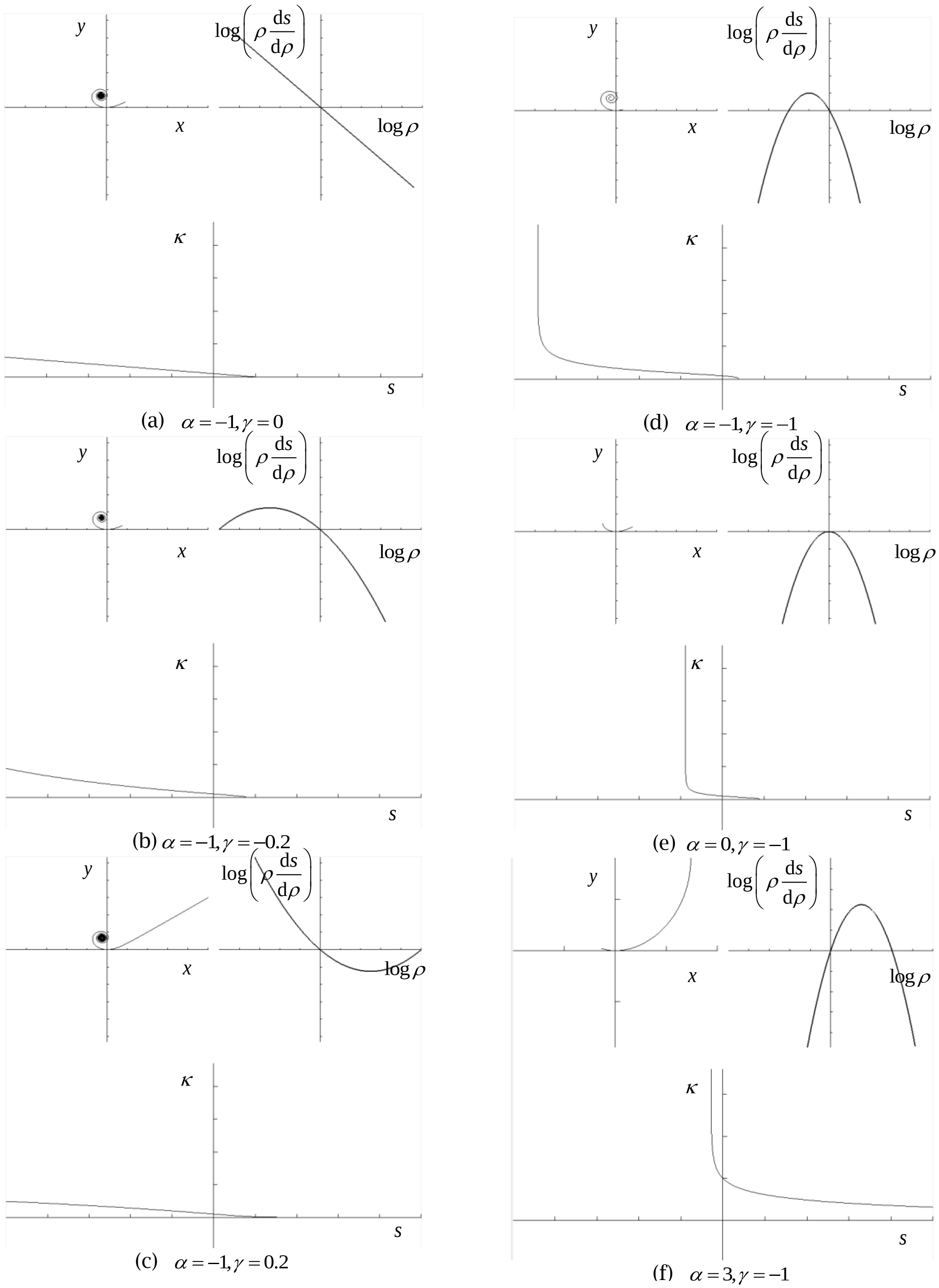
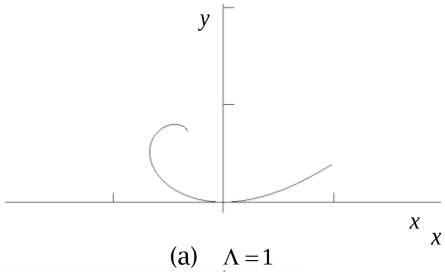
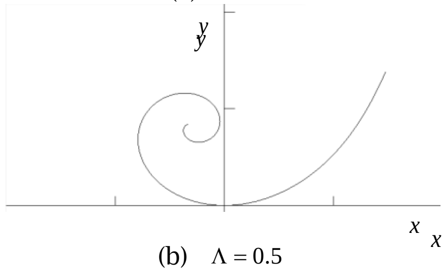


Fig. 3: Various quadratic log-aesthetic curves (upper left), logarithmic curvature graphs (right) and curvature plots (bottom).



(a) $\Lambda = 1$



(b) $\Lambda = 0.5$

Fig. 4 $\alpha = -0.2, \gamma = -0.5$

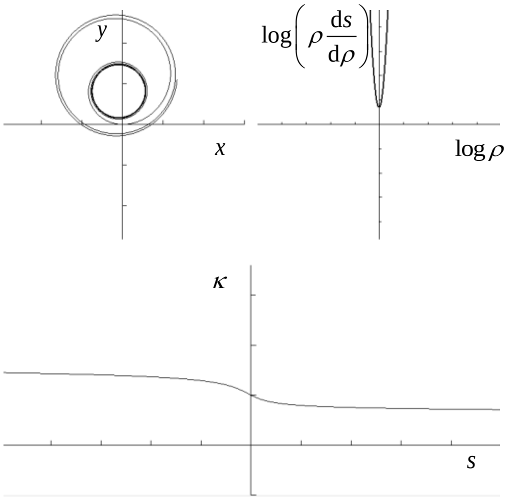


Fig. 5 $\alpha = 1, \gamma = 30, \Lambda = 0.5$

As shown in Eqn. (4.4) and (4.5), the arc length of the quadratic log-aesthetic curve is finite if $\gamma < 0$ with its curvature varying from 0 to infinity. These cases are shown in Fig. 3 (b), (d), (e) and (f). If $\gamma = 0$, quadratic log-aesthetic curves become exactly log-aesthetic curves as shown in Fig. 3 (a). If $\gamma > 0$, the arc length of the curve become infinite (Fig. 3 (c), for example). In other words, the points at $\kappa = 0$ and $\kappa = \infty$ are at infinity. Fig. 4 shows the curves with $\alpha = -0.2, \gamma = -0.5$ but with different Λs. As Λ is decreased, the arc length of the curve gets longer if $\gamma < 0$. The shape of the curve also changes if the value of Λ is changed. Fig. 5 shows an interesting example of $\alpha = 1, \gamma = 30, \Lambda = 0.5$. From the curve shape and its curvature plot, the curve gets closer to the circular arc (constant κ) as s gets close to $\pm\infty$. This fact means that G^1 interpolation algorithm of [5] does not work properly since more than one curves that fit the specified control triangle may be found. Thus, a different method is necessary and we are working with this problem.

Conclusions:

This paper proposed quadratic log-aesthetic curves by extending log-aesthetic curves so that the logarithmic curvature graphs becomes quadratic. We derived the curvature function in terms of arc length for drawing the curve and clarified the characteristics. We have implemented the proposed curve in C++ and confirmed that the curves can be generated fully in real time. Quadratic log-aesthetic curves have additional degree of freedom γ and can represent a curve segment with finite arc length and the curvature varying from 0 to ∞ if $\gamma < 0$. Future work includes more detail analysis of the characteristics of quadratic log-aesthetic curves, and the application to G^1 and G^2 Hermite interpolations. We believe that the proposed curve is promising for such interpolations.

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