Title:
**Bivariate Splines over Triangular Meshes for Freeform Surface Modeling with Sharp Feature**

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Keywords:
Freeform Surface Modeling, Bivariate Splines, Triangular Meshes, Sharp Feature

DOI: 10.14733/cadconfP.2016.131-136

Introduction:
Generalizing univariate B-splines to bivariate splines is a basic strategy to construct spline surfaces in freeform surface modeling. A popular approach is to use tensor-product. A typical example is NURBS which has been an industry standard in CAD/CAE [1]. NURBS has many good properties such as clear geometric intuition, compact representation, automatic maintenance of smoothness, analytic formula, local control, and many nice algorithms. However, due to inherent tensor-product structure, NURBS has two serious limitations: (1) NURBS does not support local refinement which is often demanded in interactive modeling and engineering simulation; and (2) NURBS has difficulty in modeling shapes of arbitrary topology. To solve the first limitation, T-splines were proposed [2,3], which allow the existence of T-junctions in the control grid of the surface definition and thus enable local refinement. To solve the second limitation, subdivision surfaces were developed [4-6], which generalize B-spline surfaces to arbitrary topology. Due to the nature of recursive subdivision process, subdivision surfaces are not widely used in CAD/CAE than in animation and game industry. Another approach to generalizing univariate B-splines to bivariate splines is by non-tensor product methods, examples of which include surfaces over triangular domains [7-9], Box-splines [10], and simplex splines [11]. In particular, Box-splines are defined over uniform grids and are still constrained by the connectivity of the grids. Simplex splines are more general than Box-splines. A well-known simplex spline is the triangular B-spline or DMS-spline [12-14], which has similar setup as B-splines but needs to explicitly add auxiliary knots. The recent generalization of univariate B-splines is bivariate splines over the so-called "Delaunay configurations" (DCB-splines) [15,16]. DCB-splines and univariate B-splines share many useful properties such as the smoothness and polynomial reproduction. They are considered to be the very promising multivariate generalization of univariate B-splines and have been successfully used in data reconstruction and visualization [17-20]. Despite their mathematical elegance, the DCB-splines do not provide an ideal user-interface for interactive modeling. In particular, the connectivity relation among the control points of a DCB-spline is ambiguous. As a result, it is not easy to determine which region of the surface will be influenced when one or some control points are manipulated. Moreover, the number of bases is unknown before the actual computation of Delaunay configuration.

In this paper, we present a novel scheme to construct bivariate spline surfaces from triangular meshes which are topologically equivalent to a disk. Triangular meshes are nowadays widely used in geometric modeling, which enables our work to be compatible with existing modeling systems and thus practically useful. The key technique behind the scheme is a set of knot selection rules that define local configurations of a triangulation, which we call the directed-one-ring-cycle (D1RC) configurations. The D1RC configurations of a triangulation actually define a bivariate spline space. Based on the concept of D1RC configurations, we define a bivariate spline surface from the input triangular mesh, which is piecewise rational. We call such surfaces D1RC-spline surfaces. The advantages of the D1RCspline surfaces are that they define the surfaces in a similar fashion of standard NURBS, i.e., the input meshes serve as the control meshes which give a rough approximation of the surfaces; there is no need to add auxiliary knots; and the resulting surfaces are $C^{k-1}$ continuous for degree $k$ surfaces. Moreover, shape features can also be modeled by simply setting special D1RC configurations. Overall, the contributions of the paper are three-folds:

- We define D1RC configurations on an arbitrary triangulation domain, which are used for knot selection. Based on the D1RC configurations, we further define D1RC-spline functions that share many properties as univariate B-splines.
- We present a novel approach to define a rational spline surface from an input triangular mesh. The surface has nice geometric properties as NURBS, which include affine invariance, local control, convex hull properties, $C^{k-1}$ continuity where $k$ is the degree of the surface, etc.
- We also provide a strategy to create sharp features in an overall smooth surface intuitively.

Main Idea and Method:

The input to our problem is a triangular mesh topologically equivalent to a disk, which can be represented as $\boldsymbol{T} = \{\boldsymbol{V}, \boldsymbol{E}, \boldsymbol{F}\}$ where $\boldsymbol{V}$ is a set of vertices, $\boldsymbol{E}$ is a set of edges and $\boldsymbol{F}$ is a set of triangles. The goal is to construct a spline surface from $\boldsymbol{T}$ such that the vertices of $\boldsymbol{T}$ serve as the control vertices as in NURBS and the triangular mesh mimics the shape of the spline surface. Our basic idea is to properly define bivariate spline functions over a 2D triangulation domain by carefully selecting knots so that these bivariate splines exhibit those useful properties of univariate B-splines, and then use them to construct the blending functions for rational spline surface definition. The main steps of our method can be outlined below:

- Parameterize the input triangular mesh
- Construction of D1RC configurations and D1RC-splines
- Construction of the rational spline surface
- Modeling of sharp features

They are further elaborated in the following subsections.

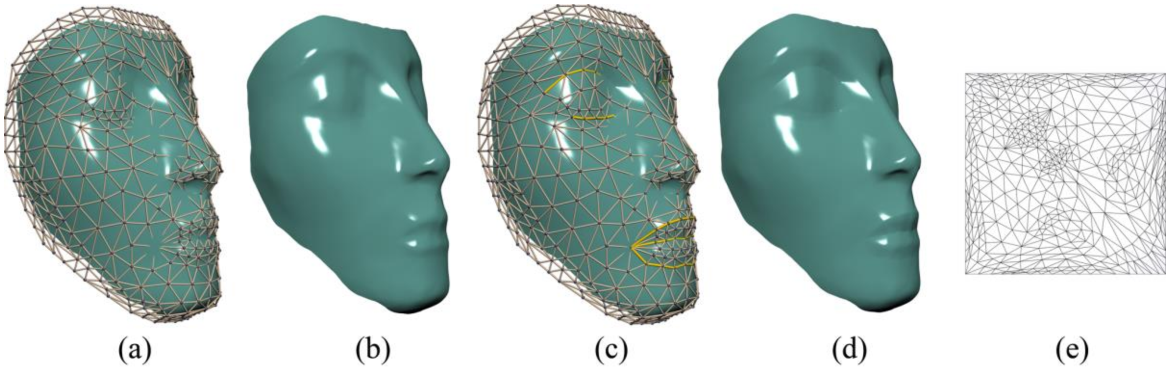


Fig. 1: Degree 3 D1RC-spline surfaces. (a) Input triangular mesh; (b) The D1RC-spline surface; (c) Specify sharp edges (in yellow) on the control mesh; (d) The resulting D1RC-spline surface with sharp features; (d) The parameterization of the input mesh in (a).

*Parameterization*

To define a parameter domain and knots for spline surfaces, we parameterize the input triangular mesh, which maps the triangular mesh to a triangulation on a 2D domain. For convenience, we denote the triangulation by $T = \{V, E, F\}$ which are the images of $\boldsymbol{T} = \{\boldsymbol{V}, \boldsymbol{E}, \boldsymbol{F}\}$. There have been many parameterization methods such as conformal parameterization and equiareal parameterization. In this paper, we use the mean value coordinates based parameterization method [21], which is a conformal parameterization and is often suggested because the one-to-one mapping is guaranteed when the domain boundary is convex. Fig.1(e) shows an example of parameterization of the input triangular mesh given in Fig.1(a).

*D1RC Configurations and D1RC-Splines*

A spline function is defined over knots. Compared to univariate B-splines, it is much more challenging to specify knots for bivariate splines over an arbitrary triangulation domain. Given a triangulation $T = \{V, E, F\}$, we define a degree $k$ directed-one-ring cycle (D1RC) configuration as $X = (X_C, X_I)$ where $X_I$ is an interior vertex set consisting of one vertex, two vertices of an edge, or three vertices of a face of $T$ corresponding to $k$ = 1, 2 and 3, respectively; and $X_C$ is the one-ring-cycle of $X_I$. The one-ring-cycle of a vertex set is formed by the vertices of the polygon, which is the union of all triangles incident to any vertex in the set, and these vertices are ordered counter-clockwise.

Let $X = (X_C, X_I)$ be a degree $k$ D1RC configuration with $X_C = (v_0,\ldots,v_n)$. A degree $k$ D1RC-spline associated with $X$ is the function:

$$b_X^k(t) = \sum_{i=0}^{n} \det(v_i, v_{i+1}, t) M(t \mid X_I \cup \{v_i, v_{i+1}\}), \quad t \in R^2 \tag{1}$$

where $\det(v_i, v_{i+1}, t)$ is the area of directed triangle formed by $v_i, v_{i+1}, t$, and $M(t \mid X_I \cup \{v_i, v_{i+1}\})$ is the simplex spline defined over the knot set $X_I \cup \{v_i, v_{i+1}\}$. See Fig.2 for a degree D1RC configuration (left) and its corresponding degree 3 D1RC-spline function (right). It can be proved that a degree $k$ D1RC spline is non-negative, of local support, and $C^{k-1}$ continuous if the knots are in generic positions.

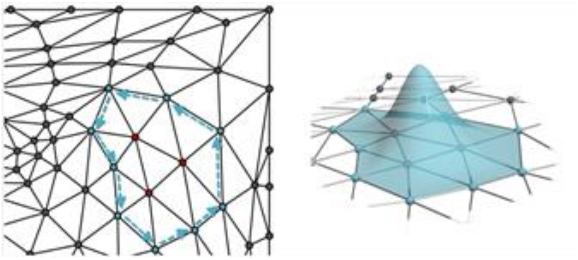

Fig. 2: A degree 3 D1RC configuration and its corresponding D1RC-spline.

*Construction of the Rational Spline Surface*

Given an input triangular mesh $\boldsymbol{T} = \{\boldsymbol{V}, \boldsymbol{E}, \boldsymbol{F}\}$ and its corresponding triangulation $T = \{V, E, F\}$ on the parameter domain, we denote by $X_T^k$ all the degree $k$ D1RC configurations and by $\Omega$ the sub-region of $T$ by removing all the triangles lying on the outermost layer. The D1RC-spline surface is defined as

$$\boldsymbol{P}(t) = \sum_{X=(X_C,X_I)\in X_T^k} \left( \frac{1}{k} \sum_{v_i \in X_I} \boldsymbol{v}_i \right) \frac{b_X^k(t)}{\sum_{X \in X_T^k} b_X^k(t)} = \sum_{\boldsymbol{v}_i \in \boldsymbol{V}_i} \boldsymbol{v}_i B_i^k(t), \quad t \in \Omega \tag{2}$$

where $\frac{1}{k}\sum_{v_i \in X_I} \boldsymbol{v}_i$ is the average of all the 3D vertices related to the D1RC configuration $X = (X_C, X_I)$, and $B_i^k(t)$ is a linear combination of normalized D1RC-spline defined in Eqn.(1), which is a piecewise rational function. In this way, the D1RC-spline surface is defined as the sum of control points multiplied by blending functions, in a fashion similar to NURBS. Fig.1 (b) shows such a spline surface.

*Modeling of Sharp Features*

The surface defined by Eqn.(2) is $C^{k-1}$ continuous in general. In order to introduce sharp features, we can use collinear knots which are an analog of multiple knots in univariate B-splines. In fact, if there are more than $s$ knots collinear, the constructed spline is $C^{k-s+1}$ continuous along the line. Upon this, we propose a special construction of D1RC configurations to achieve $C^0$ continuity along a specified edge. Assume that an edge with vertices $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are labelled as a sharp edge and their corresponding knots in $R^2$ are $v_1$ and $v_2$. We construct two degree $k$ D1RC configurations $(X_C^j, X_I^j)$ for $j$ = 1, 2, with

$X_I^1 = \{v_1 + \frac{s}{k}(v_2 - v_1) : s = 0, \cdots, k-1\}$ and $X_I^2 = \{v_2 + \frac{s}{k}(v_1 - v_2) : s = 0, \cdots, k-1\}$. Hence there are $k$+1 knots along the edge $v_1v_2$, which result in $C^0$ continuity. Refer to Fig.1(c) for an example where a few edges in the areas of the right eye and the mouse are labelled as sharp edges. This results in sharp features in the surface as shown in Fig.1(d).

Examples:
A few more examples are provided to demonstrate the capability of the proposed D1RC-spline surfaces. Fig.3 shows an example where moving one vertex of the input triangular mesh locally adjusts the shape of the surface. Fig.4 shows two degree 3 D1RC-spline surfaces. Fig.5 shows how to introduce sharp features to enhance the quality of the surface.

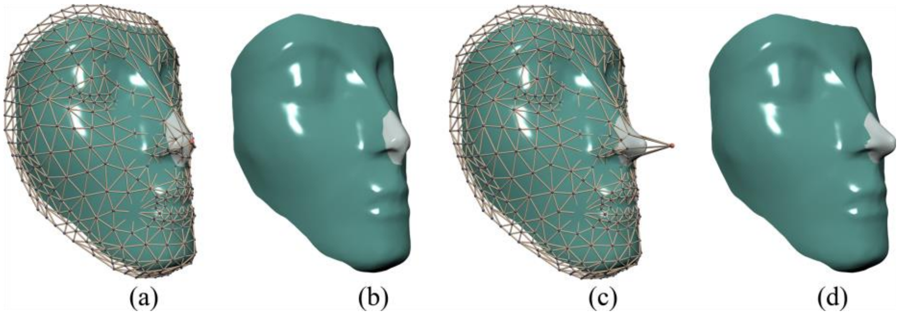


Fig. 3: The influence region of a control point in a degree 3 D1RC-spline surface. (a) The D1RC-spline surface and the control mesh, where the selected control point and his influence region are visualized in red and white, respectively; (b) The D1RC-spline surface; (c) The selected control point is moved and the shape of the surface is locally adjusted; (d) the adjusted surface.

Fig. 4: Left: input mesh with 631 vertices and the resulting D1RC-spline surface; Right: input mesh with 1290 vertices and the resulting D1RC-spline surface.

Conclusions:
We have described a novel approach to generalizing univarite B-splines to bivariate splines defined on triangular meshes. The proposed scheme defines spline surfaces in a way similar to that of NURBS, but has less restriction on the connectivity of the input mesh just as subdivision surfaces. The work provides a new way to construct bivariate splines on general triangular meshes and inspires further exploration.

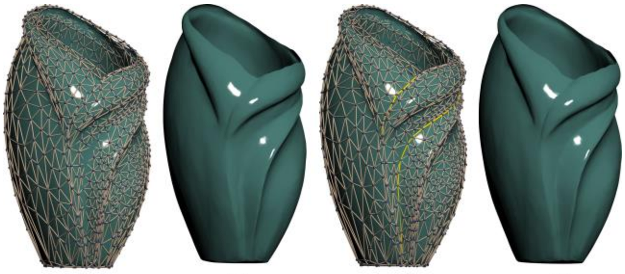


Fig. 5: (a) Input triangular mesh with 820 vertices; (b) Degree 3 D1RC-spline surface; (c) Some edges are labelled as a sharp edges (in yellow); (d) The resulting surface with $C^0$ continuity along the feature lines corresponding to the labelled edges.

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