Title:
Reformulation of Generalized Log-aesthetic Curves with Bernoulli Equations

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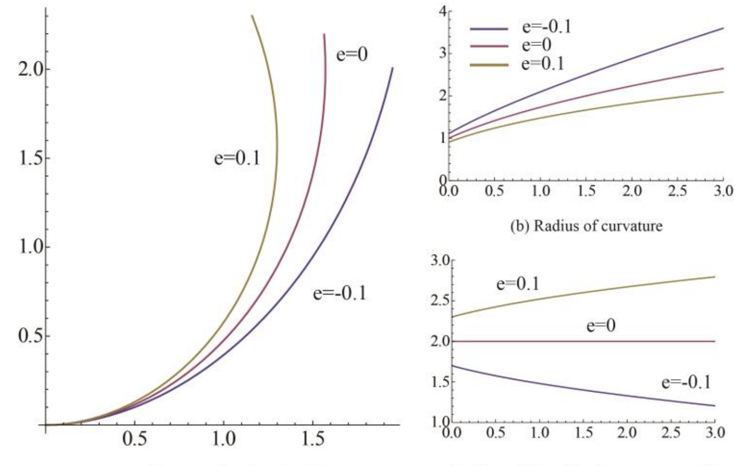
Introduction:
A Generalized Log-Aesthetic Curve (GLAC) is the general formulation of emerging the Log-Aesthetic (LA) curves for aesthetic industrial design. GLAC has an extra degree of freedom compared to LA curve which makes it versatile for design. There are two approaches employed to develop GLACs namely ρ-shift and κ-shift [2]. To note, κ-shift GLAC is a better formulation of GLAC since its directional angle can be obtained analytically as compared to ρ-shift GLAC. Figure 1 shows two examples of κ-shift GLAC with a LAC, their radius of curvature (RoC), logarithmic curvature graph (LCG) [2].

Fig. 1: Examples of κ-shift GLAC and their RoC and LCG.

Recently, Sato and Shimizu [4] reported the relationship between the fundamental equation of Log-



(a) Examples of κ–GLACs

(b) Radius of curvature

(c) Slope of logarithmic curvature graph λ

aesthetic curve and Riccati differential equations. They considered the case of ρ-shift GLAC and reported its representation in the form of Riccati equation. It is well known that solving Riccati equation involves reduction of order which is a painstaking trial and error approach to find for a solution. This paper completes the investigation by analyzing κ-shift GLAC. We derived the formula of the κ-shift GLAC as a solution of a Bernoulli equation which can be solved with various approaches.

Riccati Differential Equations:

Generally a Riccati differential equation [1] is given by

$$\frac{dy}{dx} + P(x) + Q(x)\, y + R(x)y^2 = 0 \tag{1}$$

It is known that equation (1) cannot be solved by integration in general. However if a particular solution is known, then it can be solved. This solution method is known as reduction of order which may involve trial-error approach. In the case of $P(x) = 0$, the we may convert to Bernoulli equation in which a feasible solution can be obtained by various means. Assume that a particular solution of Eq. (1) is given by $\eta(x)$ and the general solution is given by $y = \eta(x) + z(x)$. Then Eq. (1) becomes

$$\frac{dz}{dx} + \{Q(x) + 2R(x)Q(x)\,\eta\}z + R(x)z^2 = 0 \tag{2}$$

The above equation is classified as a Bernoulli equation which is discussed in the following section.

Bernoulli Differential Equations:

Let n is a constant and $n \neq 1$, a Bernoulli differential equation [1] is given by

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \tag{3}$$

If $n = 2$, equation (3) is regarded as a Riccati equation as shown in the previous section. By dividing both sides of Eq. (3) with y^n, we obtain

$$\frac{1}{y^n}\frac{dy}{dx} + P(x)\frac{1}{y^{n-1}} = Q(x) \tag{4}$$

By letting $z = 1/y^{n-1}$, we can further simplify as follows

$$\frac{1}{1-n}\frac{dz}{dx} + P(x)z = Q(x) \tag{5}$$

The equation (5) is a linear differential equation which can be solved by separation of variables.

Riccati Equations Satisfied by the Log-aesthetic Curve:

According to Sato and Shimizu [4], similarity curvature is defined as follows:

$$S := \frac{\frac{d\kappa}{ds}}{\kappa^2} = -\frac{d\rho}{ds} \tag{6}$$

where s is arc length, κ is curvature and $\rho = 1/\kappa$ is radius of curvature. The above curvature is invariant under similarity transformation. Since the arc length s is variant by similarity transformation, we may use the directional angle θ to define the curve $\gamma(\theta)$. Since $ds = \rho d\theta$, the similarity curvature is given by

$$S(\theta) = -\frac{\frac{d\rho}{d\theta}}{\rho} \tag{7}$$

A curve is said to be a log-aesthetic curve (LAC) if it has a linear logarithmic curvature graph (LCG). Thus, in 2005 Miura [3] developed the fundamental equation of LA curves by extracting curvature function from a linear LGC. The Y axis of the logarithmic curvature graph (LCG) is given by

$$Y = -\log\left(\left|\frac{d(\log\rho)}{ds}\right|\right) = X - \log(|S|) \tag{8}$$

where $X = \log\rho$. The gradient (α) of the logarithmic curvature graph is given by

$$\alpha = \frac{dY}{dX} = 1 - \frac{\frac{dS}{dX}}{S} = 1 + \frac{\frac{dS}{d\theta}}{S^2} \tag{9}$$

Therefore a LAC with shape parameter α satisfies the following Riccati equation:

$$\frac{dS}{d\theta} = (\alpha - 1)S^2 \tag{10}$$

However, we may convert it into a Bernoulli equation which can be solved with separation of variables. Its general solution is given by

$$S(\theta) = -\frac{\lambda}{(\alpha-1)\lambda\theta + 1} \tag{11}$$

We denote the right side of the above equation as $L(\alpha, \lambda; \theta)$. By solving this equation for $\rho(\theta)$, we obtain

$$\rho(\theta) = \begin{cases} e^{\lambda\theta} & (\alpha = 1) \\ ((\alpha-1)\lambda\theta + 1)^{\frac{1}{\alpha-1}} & (\alpha \neq 1) \end{cases} \tag{12}$$

The Evolutes of Log-aesthetic Curves:

For a given smooth curve $\gamma(s)$, its evolute $\sigma(s)$ is defined by

$$\sigma(s) = \gamma(s) + \rho_\gamma(s)\, n_\gamma(s) \tag{13}$$

where $\rho_\gamma(s)$ and $n_\gamma(s)$ are its radius of curvature and normal vector of curve $\gamma(s)$ respectively. Conversely $\gamma(s)$ is called an involute of $\sigma(s)$. The first derivative of $\sigma(s)$ is given by

$$\frac{d\sigma}{ds} = \frac{d\rho_\lambda}{ds} n_\gamma \tag{14}$$

The above equation indicates that the directional angle of curve $\sigma(s)$ is alligned along the normal vector of $\gamma(s)$ and its directional angle is rotated counterclockwise by 90 degrees. Note that from the above equation, parameter s is not an arc length parameter of $\sigma(s)$. Its second derivative is given by

$$\frac{d^2\sigma}{ds^2} = \frac{d^2\rho_\lambda}{ds^2} n_\gamma - \frac{1}{\rho_\gamma}\frac{d\rho_\gamma}{ds} T_\gamma \tag{15}$$

Hence

$$\frac{d\sigma}{ds} \times \frac{d^2\sigma}{ds^2} = \frac{1}{\rho_\gamma}\left(\frac{d\rho_\gamma}{ds}\right)^2 \tag{16}$$

Therefore the radius of curvature $\rho_\sigma(s)$ of curve $\sigma(s)$ is given by

$$\rho_\sigma(s) = \frac{\left|\frac{d\sigma}{ds}\right|^3}{\left|\frac{d\sigma}{ds}\times\frac{d^2\sigma}{ds^2}\right|} = \rho_\gamma \frac{d\rho_\gamma}{ds} = \frac{d\rho_\gamma}{d\theta} \tag{17}$$

Since $dS/d\theta = (d\rho_\gamma/d\theta)^2/\rho_\gamma^2 - (d^2\rho_\gamma/d\theta^2)/\rho_\gamma$, the relationship between similarity curvatures $S(\theta)$ and $T(\theta + \pi/2)$ is given by

$$\frac{dS(\theta)}{d\theta} = S(\theta)^2 - T\left(\theta + \frac{\pi}{2}\right) S(\theta) \tag{18}$$

We regard the above equation as a Bernoulli (special case of Riccati) equation which has $T(\theta + \pi/2)$ as a coefficient. When $\alpha \neq 1$, the similarity curvature $S(\theta) = L(\alpha, \lambda; \theta)$ of LAC can be rewritten as follows:

$$\frac{dS}{d\theta} = (\alpha - 1)S^2 = S^2 - \{(2-\alpha)S\}S \tag{19}$$

This equation equivalent to the Riccati equation of the involute curve whose evolute's similarity curvature is given by $T(\theta + \pi/2) = (2-\alpha)L(\alpha, \lambda; \theta)$. Hence if $\alpha = 2$, the evolute of the LAC with $S(\theta) = L(\alpha, \lambda; \theta)$ is a circular arc ($T(\theta) = 0$). If $\alpha \neq \{1, 2\}$, the similarity curvature of the evolute is

$$T(\theta) = (2-\alpha)L\left(\alpha, \lambda; \theta - \frac{\pi}{2}\right) = L\left(\frac{1}{2-\alpha}, (2-\alpha)\lambda; \theta - \frac{\pi}{2}\right) \tag{20}$$

These results conform to those obtained by Yoshida and Saito [5].

Conversely, we assume that $\alpha \neq \{1, 2\}$ and $T(\theta) = L\left(\frac{1}{2-\alpha}, (2-\alpha)\lambda; \theta - \frac{\pi}{2}\right)$. Then LAC is a particular solution of the following Riccati equation:

$$\frac{dS}{d\theta} = S(\theta)^2 - T\left(\theta + \frac{\pi}{2}\right) S(\theta) \tag{21}$$

The general solution is obtained using the above particular solution as follows

$$S(\theta) = \frac{L(\alpha, \lambda; \theta)}{1 + C((\alpha - 1)\lambda\theta + 1)^{\frac{1}{1-\alpha}}} \tag{22}$$

where C is a constant of integration. This gives the similarity curvature of the ρ-shift GLAC [2] as shown in the next section.

<u>Similarity Curvature of $\boldsymbol{\rho}$-shift GLAC:</u>

The radius of curvature (ρ) of ρ-shift GLAC is given by

$$\rho_{\rho-GLAC}(\theta) = \begin{cases} e^{\lambda\theta} + \nu & (\alpha = 1) \\ ((\alpha\kappa - 1)\lambda\theta + 1)^{\frac{1}{\alpha-1}} + \nu & (\alpha \neq 1) \end{cases} \tag{23}$$

When $\alpha \neq \{1, 2\}$, its similarity curvature is given by

$$S_{\rho-GLAC} = -\frac{\frac{d\rho_{\rho-GLAC}}{d\theta}}{\rho_{\rho-GLAC}} = \frac{L(\alpha, \lambda; \theta)}{1 + \nu((\alpha-1)\lambda\theta + 1)^{\frac{1}{1-\alpha}}} \tag{24}$$

The above equation is equivalent to Eq. (22).

<u>$\boldsymbol{\kappa}$-shift GLAC:</u>

As stated above, there are two types of GLAC [2]; ρ-shift and κ-shift. This section shows the derivation of κ-shift GLAC.

Reciprocal of Similarity Curvature

If the similarity curvature S is invariant under similarity transformation, its reciprocal $1/S$ is also invariant if $\neq 0$. We define $V = 1/S$ and call it similarity radius of curvature since it is the reciprocal of similarity curvature.

As $dV/d\theta = -(dS/d\theta)/S^2$, Eq. (10) is written with V by

$$\frac{dV}{d\theta} = 1 - \alpha \tag{25}$$

The above differential equation can be solved analytically and we may obtain the following general solution:

$$V(\theta) = -\frac{(\alpha-1)\lambda\theta+1}{\lambda} \tag{26}$$

This satisfies $V(\theta) = 1/S(\theta)$ where $S(\theta)$ is given by Eq. (11). We denote this general solution as $M(\alpha, \lambda; \theta)$.

Derivation of ***κ****-shift GLAC Formula*

Substituting $S(\theta)$ with $V(\theta)$, Eq. (18) becomes

$$\frac{dV(\theta)}{d\theta} = T\left(\theta + \frac{\pi}{2}\right)V(\theta) - 1 \tag{27}$$

The general solution of the above equation is given by

$$V(\theta) = -\frac{((\alpha-1)\lambda\theta+1)^{\frac{\alpha-2}{\alpha-1}}\left(C+((\alpha-1)\lambda\theta+1)^{\frac{1}{\alpha-1}}\right)}{\lambda} \tag{28}$$

$$= \frac{1+C((\alpha-1)\lambda\theta+1)^{\frac{1}{1-\alpha}}}{L(\alpha,\lambda;\theta)} \tag{29}$$

The above equation is indeed the reciprocal of $S(\theta)$ given in Eq. (24).

On the other hand, the similarity radius of curvature of the κ-shift GLAC whose curvature is equal to $((\alpha-1)\lambda\theta+1)^{\frac{1}{1-\alpha}} + \nu$ is given by

$$V_{\kappa-GALC}(\theta) = -\frac{((\alpha-1)\lambda\theta+1)^{\frac{\alpha}{\alpha-1}}\left(\nu+((\alpha-1)\lambda\theta+1)^{\frac{1}{1-\alpha}}\right)}{\lambda} \tag{30}$$

Eqs. (28) and (30) may look different. However, if we define $\beta = 2-\alpha$, and substitute it into Eq.(28), we obtain

$$V(\theta) = -\frac{((1-\beta)\lambda\theta+1)^{\frac{\beta}{\beta-1}}\left(C+((1-\beta)\lambda\theta+1)^{\frac{1}{1-\beta}}\right)}{\lambda} \tag{31}$$

Hence

$$V(-\theta) = -\frac{((\beta-1)\lambda\theta+1)^{\frac{\beta}{\beta-1}}\left(C+((\beta-1)\lambda\theta+1)^{\frac{1}{1-\beta}}\right)}{\lambda} = M(\beta,\lambda;-\theta)\left(1+\nu((1-\beta)\lambda\theta+1)^{\frac{1}{\beta-1}}\right) \tag{31}$$

Therefore the solutions represent a κ-shift GLAC whose LCG gradient equals to $\beta = 2-\alpha$ and the curve direction is reversed. As expected, κ-shift GLAC reduces to $V(\theta) = M(\beta, \lambda; \theta)$ when $\nu = 0$.

Conclusions:

Sato and Shimizu has shown the relationship between LAC and ρ-shift GLAC using the concept of similarity geometry and Riccati equation. In this paper, we complete the research rewriting the fundamental formula of κ-shift GLAC as a solution of a Bernoulli equation which complies with the results obtained in previous studies.

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References:

[1] Differential equations, University of Tokyo, 1960.

[2] Gobithaasan, R.U.; Miura, K.T.: Aesthetic spiral for design, Sains Malaysiana, 40(11), 2011, 1301-1305.

[3] Miura, K.T.: A general equation of aesthetic curves and its self-affinity, Computer-Aided Design and Applications, 3(1-4), 2006, 457-464. http://dx.doi.org/10.1080/16864360.2006.10738484

[4] Sato, M.; Shimizu, Y.: The log-aesthetic curve and Riccati equations from the viewpoint of similarity geometry, Proc. JSIAM2014, 2014.

[5] Yoshida, N.; Saito, T.: Interactive aesthetic curve segments, The Visual Computer (Pacific Graphics), 22(9-11), 2006, 896-905. http://dx.doi.org/10.1007/s00371-006-0076-5